

# ADM-like Hamiltonian formulation of gravity in the teleparallel geometry

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## Abstract

Teleparallel Equivalent of General Relativity is considered as a theory of cotetrad fields on a spacetime. The  $3+1$  decomposition of the cotetrad is described by means of the lapse and the shift appearing in the ADM Hamiltonian formulation of General Relativity. We carry out the Legendre transformation and derive a Hamiltonian and a constraint algebra.

## 1 Introduction

General Relativity (GR) was formulated originally [1] as a theory describing the gravitational field by means of the pseudo-Riemannian geometry. In this formulation the main object is a spacetime metric which generates the Levi-Civita connection which is torsion-free and of non-zero curvature. Alternatively, GR can be described in terms of the teleparallel geometry of Weitzenböck [2]—here the main object is a cotetrad field on the spacetime which generates a curvature-free connection of non-zero torsion. The relation between the two formulations is established by assuming that the cotetrad in the Weitzenböck geometry is orthonormal with respect to the metric of the original formulation of GR.

This alternative description called commonly Teleparallel Equivalent of GR (TEGR) is formulated as a Lagrangian field theory (see e.g. [3, 4, 5, 6, 7] and references therein). On the other hand it is always interesting to see how a theory looks like in a Hamiltonian formulation hence there is nowadays a variety of different Hamiltonian formulations of TEGR [5, 8, 9, 10, 11, 12].

In this paper we present another Hamiltonian description of TEGR. Basic features of this description can be summarized as follows:

1. we treat TEGR as a theory of cotetrad fields i.e. of collections  $(\theta^A)$  ( $A = 0, 1, 2, 3$ ) of one-forms on a spacetime and describe its dynamics by the following action [6, 7, 13, 14]:

$$S[\theta^A] = \int -\frac{1}{2}(d\theta^A \wedge \theta_B) \wedge \star(d\theta^B \wedge \theta_A) + \frac{1}{4}(d\theta^A \wedge \theta_A) \wedge \star(d\theta^B \wedge \theta_B), \quad (1.1)$$

where  $d$  is the exterior derivative of differential forms on a manifold modeling the spacetime and  $\star$  is the Hodge operator defined by a Lorentzian metric given by the cotetrad  $(\theta^A)$ .

2. carrying out the 3 + 1 decomposition we follow the ADM Hamiltonian formulation of GR [15] and parameterize non-dynamical degrees of freedom (that is, the time components of  $(\theta^A)$ ) by means of the lapse function and the shift vector field.
3. a special kind of canonical formalism [16, 5] adapted to differential forms is used.

The main results of this paper is a Hamiltonian, a complete set of constraints on the phase space and a constraint algebra.

The paper is organized as follows: after preliminaries (Section 2) we present in Section 3 the main results of the paper and a short discussion—we decided to place the results before their derivation because the derivation involves very long, complicated and rather boring calculations. Next, in Section 4 we carry out the Legendre transformation and arrive at the Hamiltonian and the constraints. Finally in Section 5 we derive the constraints algebra.

## 2 Preliminaries

Let  $\mathbb{M}$  be a four-dimensional oriented vector space equipped with a scalar product  $\eta$  of signature  $(-, +, +, +)$ . We fix an orthonormal basis  $(v_A)$  ( $A = 0, 1, 2, 3$ ) such that the components  $(\eta_{AB})$  of  $\eta$  given by the basis form a matrix  $\text{diag}(-1, 1, 1, 1)$ . The matrix  $(\eta_{AB})$  and its inverse  $(\eta^{AB})$  will be used to, respectively, lower and raise capital Latin letter indices.

Let  $\mathcal{M}$  be a four-dimensional oriented manifold. We assume that there exists a smooth map  $\theta : T\mathcal{M} \rightarrow \mathbb{M}$  such that for every  $y \in \mathcal{M}$  the restriction of  $\theta$  to the tangent space  $T_y\mathcal{M}$  is a linear isomorphism between the tangent space and  $\mathbb{M}$  which preserves the orientations. The map  $\theta$  can be expressed by means of the orthogonal basis  $(v_A)$  as

$$\theta = \theta^A \otimes v_A,$$

where  $(\theta^A)$  are one-forms on  $\mathcal{M}$ . It is clear that the one-forms  $(\theta^A)$  form a *coreper* or a *cotetrad field* on the manifold. If  $(x^\mu)$ ,  $(\mu = 0, 1, 2, 3)$ , is a local coordinate frame on  $\mathcal{M}$  compatible with its orientation then the determinant of the matrix  $(\theta_\mu^A)$  built from the components of the forms  $\theta^A$  in the coordinate frame is positive:

$$\det(\theta_\mu^A) > 0. \quad (2.1)$$

The map  $\theta$  can be used to pull back the scalar product  $\eta$  on  $\mathbb{M}$  to the manifold  $\mathcal{M}$  turning thereby the manifold into a *spacetime*. We will denote the resulting Lorentzian metric by  $g$ ,

$$g := \eta_{AB} \theta^A \otimes \theta^B. \quad (2.2)$$

The metric  $g$  defines a volume form  $\epsilon$  on  $\mathcal{M}$  and the Hodge dual operator  $\star$  mapping differential  $k$ -forms to  $(4 - k)$ -forms on the manifold ( $k = 0, 1, 2, 3, 4$ ).

## 2.1 TEGR

In this paper we will treat TEGR as a theory of cotetrad fields on  $\mathcal{M}$  which means that the configuration space of the theory will be a set of all the maps  $\theta$  which satisfy the assumptions listed above. We choose the action (1.1) as one describing the dynamics of TEGR (for different but equivalent actions see e.g. [3, 4, 5]). Let us emphasize that the Hodge operator  $\star$  appearing in (1.1) is given by the metric (2.2) and therefore it is a function of  $(\theta^A)$ .

The passage from the action (1.1) to a Hamiltonian formulation requires as its first step a  $3 + 1$  decomposition of: the manifold  $\mathcal{M}$ , differential forms on it and a coreper  $(\theta^A)$ .

## 2.2 $3 + 1$ decomposition of $\mathcal{M}$

To carry out a  $3 + 1$  decomposition of the action (1.1) we have to impose some additional assumptions on the manifold  $\mathcal{M}$  and the map  $\theta$ . We require that

1.  $\mathcal{M} = \mathbb{R} \times \Sigma$ , where  $\Sigma$  is a three-dimensional manifold.

This assumption allows us to introduce a family of curves in  $\mathcal{M}$  parameterized by points of  $\Sigma$ —given  $x \in \Sigma$  we define

$$\mathbb{R} \ni t \mapsto (t, x) \in \mathbb{R} \times \Sigma = \mathcal{M}. \quad (2.3)$$

These curves generates a global vector field on  $\mathcal{M}$  which will be denoted by  $\partial_t$ . We require moreover that

2. the map  $\theta$  is such that  $\partial_t$  is timelike with respect to the metric  $g$  defined by  $\theta$ .

3. the map  $\theta$  is such that for every  $t \in \mathbb{R}$  the submanifold  $\Sigma_t := \{t\} \times \Sigma$  is spatial with respect to  $g$ .

In order to not be troubled by boundary terms in the Hamiltonian formulation we assume that

4.  $\Sigma$  is a compact manifold without boundary.

Assumption 1 allows us to define a function on  $\mathcal{M}$  which maps a point  $y$  to a number  $t$  such that  $y \in \Sigma_t$ . Abusing the notation we will use the letter  $t$  to denote the function. Let  $(x^i)$ ,  $(i = 1, 2, 3)$ , be local coordinates on  $\Sigma$ . The coordinates together with the function  $t$  define local coordinates on  $\mathcal{M}$  which associate with an appropriate  $y \in \Sigma_t$  four numbers  $(x^0 \equiv t, x^i) \equiv (x^\mu)$ . Obviously, on the domain of such a coordinate frame the vector field  $\partial_0$  given by the frame coincides with  $\partial_t$  generated by the curves (2.3). Since now we will restrict ourselves to coordinate frames  $(x^\mu)$  on  $\mathcal{M}$  of this sort assuming additionally that each frame we are going to use is compatible with the orientation of the manifold.

Note that these coordinate frames induce an orientation of  $\Sigma$  which since now will be treated as an *oriented* manifold.

Let us finally emphasize that in this paper the spacetime indices will be denoted by lower case Greek letters and will range from 0 to 3 and the spatial indices will be denoted by lower case Latin letters and will range from 1 to 3.

### 2.3 Decomposition of differential forms

Denote by  $\boldsymbol{d}$  the exterior derivative of forms on  $\mathcal{M}$  and by  $d$  the exterior derivative of forms on  $\Sigma$ . A  $k$ -form  $\alpha$  on  $\mathcal{M}$  can be decomposed with respect to the decomposition  $\mathcal{M} = \mathbb{R} \times \Sigma$  as follows [16, 5]

$$\alpha = {}^\perp\alpha + \underline{\alpha}$$

where

$${}^\perp\alpha := \boldsymbol{d}t \wedge \alpha_\perp \quad \text{with} \quad \alpha_\perp := \partial_t \lrcorner \alpha$$

is its “timelike” part and

$$\underline{\alpha} := \partial_t \lrcorner (\boldsymbol{d}t \wedge \alpha)$$

its “spatial” part.

$\underline{\alpha}$  is a form on  $\mathcal{M}$  which can be expressed in a coordinate frame  $(t, x^i)$  as

$$\underline{\alpha} = \frac{1}{k!} \alpha_{i_1 \dots i_k}(t, x^i) \boldsymbol{d}x^{i_1} \wedge \dots \wedge \boldsymbol{d}x^{i_k}.$$

The form naturally defines a form on  $\Sigma$  (or more precisely, a one parameter family of forms on  $\Sigma$  the parameter being the coordinate  $t$ )

$$\underline{\alpha}' := \frac{1}{k!} \alpha_{i_1 \dots i_k}(t, x^i) dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

Moreover, it is possible to restore the original form  $\underline{\alpha}$  from  $\underline{\alpha}'$ : given the latter one we define

$$\partial_t \lrcorner \underline{\alpha} := 0, \quad \underline{\alpha}(\vec{X}_1, \dots, \vec{X}_k) := \underline{\alpha}'(\vec{X}_1, \dots, \vec{X}_k)$$

for all vector fields  $(\vec{X}_1, \dots, \vec{X}_k)$  tangent to the foliation  $\{\Sigma_t\}$  of  $\mathcal{M}$ . Therefore in the sequel we will not distinguish between  $\underline{\alpha}$  and  $\underline{\alpha}'$ . There is however one subtlety concerning Lie derivatives of  $\underline{\alpha}$  and  $\underline{\alpha}'$ . Let  $\vec{X}$  be a vector field on  $\mathcal{M}$  tangent to the foliation  $\{\Sigma_t\}$ . Denote by  $\mathcal{L}_{\vec{X}}$  the Lie derivative on  $\mathcal{M}$  with respect to  $\vec{X}$  and by  $\mathcal{L}'_{\vec{X}}$  the Lie derivative on  $\Sigma_t$  with respect to  $\vec{X}$  restricted to the submanifold. Then in general  $\mathcal{L}_{\vec{X}} \underline{\alpha}$  cannot be identified with  $\mathcal{L}'_{\vec{X}} \underline{\alpha}'$ . Indeed, if  $\alpha$  is a one-form on  $\mathcal{M}$  then

$$\mathcal{L}_{\vec{N}} \underline{\alpha} = (N^\mu \partial_\mu \underline{\alpha}_\nu + \underline{\alpha}_\mu \partial_\nu N^\mu) dx^\mu = \underline{\alpha}_i \partial_0 N^i dt + (N^i \partial_i \underline{\alpha}_j + \underline{\alpha}_i \partial_j N^i) dx^j.$$

and only the last term in this equation can be identified with  $\mathcal{L}'_{\vec{X}} \underline{\alpha}'$ . However, in the sequel we will never encounter Lie derivatives  $\mathcal{L}_{\vec{X}} \underline{\alpha}$  as defined above but we will do encounter derivatives  $\mathcal{L}'_{\vec{X}} \underline{\alpha}'$ . Since we would like our notation to be as simple as possible since now we will use the symbol  $\mathcal{L}_{\vec{X}} \underline{\alpha}$  to denote the derivative  $\mathcal{L}'_{\vec{X}} \underline{\alpha}'$ .

Similarly,  $\alpha_\perp$  is a form on  $\mathcal{M}$ , but it can be treated as a (one parameter family of) form(s) on  $\Sigma$ .

Basic properties of the maps  $\alpha \mapsto \alpha^\perp$  and  $\alpha \mapsto \underline{\alpha}$  (here  $\alpha$  is a  $k$ -form) [16, 5]:

$$\begin{aligned} {}^\perp({}^\perp \alpha) &= {}^\perp \alpha, & \underline{({}^\perp \alpha)} &= {}^\perp(\underline{\alpha}) = 0, & \underline{(\underline{\alpha})} &= \underline{\alpha}, \\ \underline{(\alpha \wedge \beta)} &= \underline{\alpha} \wedge \underline{\beta}, & {}^\perp(\alpha \wedge \beta) &= ({}^\perp \alpha) \wedge \underline{\beta} + \underline{\alpha} \wedge ({}^\perp \beta), & \underline{\alpha_\perp} &= \alpha_\perp, \\ \partial_t \lrcorner \underline{\alpha} &= 0, & (\alpha \wedge \beta)_\perp &= \alpha_\perp \wedge \underline{\beta} + (-1)^k \underline{\alpha} \wedge \beta_\perp, & \underline{(d\alpha)} &= d\underline{\alpha}, \\ (d\alpha)_\perp &= \mathcal{L}_{\partial_t} \underline{\alpha} - d\alpha_\perp, & d\alpha &= dt \wedge \mathcal{L}_{\partial_t} \underline{\alpha} - dt \wedge d\alpha_\perp + d\underline{\alpha}, \end{aligned} \tag{2.4}$$

where  $\mathcal{L}_{\partial_t}$  denotes the Lie derivative with respect to the vector field  $\partial_t$ .

## 2.4 Decomposition of the coreper

Since each  $\theta^A$  is a one-form it decomposes as

$$\theta^A = \theta_\perp^A dt + \underline{\theta}^A. \tag{2.5}$$

It turns out that  $\theta_\perp^A$  is a function of  $\underline{\theta}^A$  and some additional parameters [9, 17]:

$$\theta_\perp^A = N\xi^A + \vec{N} \lrcorner \underline{\theta}^A, \quad (2.6)$$

$$\xi^A := -\frac{1}{3!} \varepsilon^A{}_{BCD} * (\underline{\theta}^B \wedge \underline{\theta}^C \wedge \underline{\theta}^D), \quad (2.7)$$

$$N > 0, \quad (2.8)$$

where

1.  $N$  is a function on  $\mathcal{M}$  called *lapse*;
2.  $\vec{N}$  is a vector field on  $\mathcal{M}$  called *shift*. It is tangent at each point to a submanifold  $\Sigma_t$  passing through the point—in an admissible coordinate system  $(t, x^i)$

$$\vec{N} = N^i \partial_i;$$

3.  $\varepsilon_{ABCD}$  is a volume form on  $\mathbb{M}$  given by the scalar product  $\eta$ ;
4.  $*$  is the Hodge operator on  $\Sigma_t$  given by an Euclidean metric  $q$  on  $\Sigma_t$  defined as the restriction of  $g$  to the submanifold:

$$\begin{aligned} q &= q_{ij} dx^i \otimes dx^j := g_{ij} dx^i \otimes dx^j = \eta_{AB} \underline{\theta}^A \otimes \underline{\theta}^B, \\ q_{ij} &= \eta_{AB} \theta_i^A \theta_j^B. \end{aligned} \quad (2.9)$$

The functions  $\xi^A$  satisfy the following important conditions [9]:

$$\xi^A \xi_A = -1, \quad \xi^A \underline{\theta}_A = 0. \quad (2.10)$$

These two equations imply

$$\xi^A d\xi_A = 0, \quad d\xi^A \wedge \underline{\theta}_A + \xi^A d\underline{\theta}_A = 0. \quad (2.11)$$

Fixing the value of the index  $\mu$  we can treat the four components  $\theta_\mu^A$  as a function on  $\mathcal{M}$  valued in  $\mathbb{M}$ . The conditions (2.10) mean that for every  $y \in \mathcal{M}$  the vectors  $(\xi^A(y), \theta_i^A(y))$  form a basis of  $\mathbb{M}$ .

The decomposition of the coreper allows us to change the way we parameterize the subspace of the configuration space consisting of the corepers  $(\theta^A)$  satisfying the assumptions listed in Subsection 2.2—instead of  $(\theta_\perp^A, \underline{\theta}^A)$  we will use  $(N, \vec{N}, \underline{\theta}^A)$  as parameters on the subspace. This change is obviously motivated by our wish to obtain an ADM-like Hamiltonian formulation of TEGR and can be seen as a source of difference between this approach and that of [11] and [12].

## 2.5 Decomposition of the spacetime metric

Setting to (2.2) the coreper  $(\theta^A)$  decomposed according to (2.5) and (2.6) we obtain the standard 3 + 1 decomposition of the spacetime metric  $g$  [15]:

$$g = (-N^2 + N^i N^j q_{ij}) dt^2 + 2N^i q_{ij} dt dx^j + q, \quad (2.12)$$

where  $q$  given by (2.9) is the Euclidean metric induced on  $\Sigma$ . This decomposition justifies calling the function  $N$  the lapse and the vector field  $\vec{N}$  the shift (for a more precise justification see [17]).

The metric  $q$  and its inverse  $q^{-1}$ ,

$$q^{-1} := q^{ij} \partial_i \otimes \partial_j, \quad q^{ij} q_{jk} = \delta^i_k, \quad (2.13)$$

will be used to, respectively, lower and raise, indices (here: lower case Latin letters) of components of tensor fields defined on  $\Sigma$ . In particular we will often map one-forms to vector fields on  $\Sigma$ —a vector field corresponding to a one form  $\alpha$  will be denoted by  $\vec{\alpha}$  i.e. if  $\alpha = \alpha_i dx^i$  then

$$\vec{\alpha} := q^{ij} \alpha_i \partial_j.$$

The metric  $q$  defines a volume form  $\epsilon$  on  $\Sigma$  and the Hodge operator  $*$  acting on differential forms on the manifold.

Let us emphasize finally that (as it follows from (2.9)) the metric  $q$  can be defined explicitly in terms of the restricted forms  $(\underline{\theta}^A)$ . Therefore all object defined by  $q$  (as  $q^{-1}$ ,  $\epsilon$  and  $*$ ) are in fact functions of  $(\underline{\theta}^A)$ .

## 3 Main results

In this section we are going to present the main results of the paper: the Hamiltonian and the constraint algebra for TEGR defined by the action (1.1). Let us emphasize that to describe the canonical framework of TEGR we will use a Hamiltonian formalism adapted to differential forms [16, 5] (see also [17]).

Before we will show the results let us simplify the notation—since now we will denote the “spatial” part of the one-form  $\theta^A$  by  $\theta^A$ , i.e.

$$\underline{\theta}^A \equiv \theta^A \quad (3.1)$$

and its Lie derivative with respect to  $\partial_t$  by  $\dot{\theta}^A$  i.e.

$$\mathcal{L}_{\partial_t} \underline{\theta}^A \equiv \dot{\theta}^A. \quad (3.2)$$

### 3.1 Hamiltonian and constraints

In the action (1.1) there is no Lie derivative with respect to  $\partial_t$  of the lapse  $N$  and the shift  $\vec{N}$  but there is one of  $\theta^A$ . Therefore the two former variables are treated as Lagrange multipliers, while the latter one becomes one of the canonical variables. A point in the phase space of the theory consists of

1. a quadruplet of one-forms  $(\theta^A)$  on  $\Sigma$  such that at each point  $x \in \Sigma$  the rank of the matrix  $(\theta_i^A(x))$  is maximal;
2. the momentum  $(p_A)$  conjugate to  $\theta^A$ : since  $\Sigma$  is three dimensional and  $\theta^A$  is a one-form  $(p_A)$  is a quadruplet of two-forms.

Equivalently, a point in the phase space of the theory consists of

1. a map  $\theta : T\Sigma \rightarrow \mathbb{M}$  such that for every  $x \in \Sigma$  the restriction of  $\theta$  to  $T_x\Sigma$  is a linear injection;
2. the momentum  $p$  as a two-form on  $\Sigma$  valued in  $\mathbb{M}^*$  being the dual space to  $\mathbb{M}$ .

The Legendre transformation is given by<sup>1</sup>

$$p_A = \frac{\partial L_\perp}{\partial \dot{\theta}^A}, \quad (3.3)$$

where  $L$  is the integrand in (1.1). The momentum turns out to be quite complicated function of the variables  $N, \vec{N}, \theta^A$  and  $\dot{\theta}^A$ :

$$p_A = N^{-1} \left( \theta_B \wedge *[\dot{\theta}^B \wedge \theta_A - \xi^B dN \wedge \theta_A - N(d\xi^B \wedge \theta_A - d\theta^B \wedge \xi_A) - \mathcal{L}_{\vec{N}}\theta^B \wedge \theta_A] - \frac{1}{2} \theta_A \wedge *[\dot{\theta}^B \wedge \theta_B - N(d\xi^B \wedge \theta_B - d\theta^B \wedge \xi_B) - \mathcal{L}_{\vec{N}}\theta^B \wedge \theta_B] \right), \quad (3.4)$$

where  $\mathcal{L}_{\vec{N}}$  denotes the Lie derivative on  $\Sigma_t$  with respect to  $\vec{N}$ .

The Legendre transformation is not invertible and one encounters the following *primary constraints*

$$\theta^A \wedge *d\theta_A + \xi^A p_A = 0, \quad (3.5)$$

$$\theta^A \wedge *p_A - \xi^A d\theta_A = 0 \quad (3.6)$$

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<sup>1</sup>For a definition of the partial derivative  $\partial L_\perp / \partial \dot{\theta}^A$  see [17].



called here *boost* and *rotation* constraints respectively (for a justification of the names see Subsection 3.3.1). Their smeared versions read

$$B(a) := \int_{\Sigma} a \wedge (\theta^A \wedge *d\theta_A + \xi^A p_A), \quad (3.7)$$

$$R(b) := \int_{\Sigma} b \wedge (\theta^A \wedge *p_A - \xi^A d\theta_A), \quad (3.8)$$

where  $a$  and  $b$  are one-forms on  $\Sigma$ .

The Hamiltonian

$$H_0 := \int_{\Sigma} \dot{\theta}^A \wedge p_A - L_{\perp}$$

is unambiguously defined on the subset of the phase space distinguished by vanishing of the primary constraints and is of the following form

$$\begin{aligned} H_0[\theta^A, p_B, N, \vec{N}] = & \int_{\Sigma} N \left( \frac{1}{2} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) - \frac{1}{4} (p_A \wedge \theta^A) \wedge * (p_B \wedge \theta^B) - \right. \\ & - \xi^A \wedge dp_A + \frac{1}{2} (d\theta_A \wedge \theta^B) \wedge * (d\theta_B \wedge \theta^A) - \frac{1}{4} (d\theta_A \wedge \theta^A) \wedge * (d\theta_B \wedge \theta^B) \Big) - \\ & - d\theta^A \wedge (\vec{N} \lrcorner p_A) - (\vec{N} \lrcorner \theta^A) \wedge dp_A. \end{aligned} \quad (3.9)$$

It can be extended to the whole phase space by adding the primary constraints:

$$H[\theta^A, p_B, N, \vec{N}, a, b] = H_0[\theta^A, p_B, N, \vec{N}] + B(a) + R(b), \quad (3.10)$$

where the one-forms  $a$  and  $b$  play the role of Lagrange multipliers.

The Lagrange multipliers  $N$  and  $\vec{N}$  appearing in the Hamiltonian (3.10) generate the following *secondary constraints*

$$\begin{aligned} & \frac{1}{2} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) - \frac{1}{4} (p_A \wedge \theta^A) \wedge * (p_B \wedge \theta^B) - \xi^A \wedge dp_A + \\ & + \frac{1}{2} (d\theta_A \wedge \theta^B) \wedge * (d\theta_B \wedge \theta^A) - \frac{1}{4} (d\theta_A \wedge \theta^A) \wedge * (d\theta_B \wedge \theta^B) = 0, \\ & - d\theta^A \wedge (\partial_i \lrcorner p_A) - (\partial_i \lrcorner \theta^A) \wedge dp_A = 0. \end{aligned} \quad (3.11)$$

called *scalar* and *vector* constraints respectively. Smeared versions of the constraints read

$$\begin{aligned} S(M) := & \int_{\Sigma} M \left( \frac{1}{2} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) - \frac{1}{4} (p_A \wedge \theta^A) \wedge * (p_B \wedge \theta^B) - \xi^A \wedge dp_A + \right. \\ & \left. + \frac{1}{2} (d\theta_A \wedge \theta^B) \wedge * (d\theta_B \wedge \theta^A) - \frac{1}{4} (d\theta_A \wedge \theta^A) \wedge * (d\theta_B \wedge \theta^B) \right), \end{aligned} \quad (3.12)$$

$$V(\vec{M}) := \int_{\Sigma} -d\theta^A \wedge (\vec{M} \lrcorner p_A) - (\vec{M} \lrcorner \theta^A) \wedge dp_A, \quad (3.13)$$

where  $M$  is a function on  $\Sigma$  and  $\vec{M}$  a vector field on the manifold.

The Hamiltonian  $H_0$  is a sum of the smeared scalar and vector constraints,

$$H_0[\theta^A, p_B, N, \vec{N}] = S(N) + V(\vec{N}), \quad (3.14)$$

and the extended Hamiltonian is a sum of all the constraints:

$$H[\theta^A, p_B, N, \vec{N}, a, b] = S(N) + V(\vec{N}) + B(a) + R(b). \quad (3.15)$$

### 3.2 Constraint algebra

Let us now present the algebra of constraints. The Poisson brackets of the smeared boosts and rotation constraints read:

$$\begin{aligned} \{B(a), B(a')\} &= -R(* (a \wedge a')), \\ \{R(b), R(b')\} &= R(* (b \wedge b')), \\ \{B(a), R(b)\} &= B(* (a \wedge b)). \end{aligned} \quad (3.16)$$

The bracket of the scalar constraints is most complex:

$$\begin{aligned} \{S(M), S(M')\} &= V(\vec{m}) + B\left(\theta^B * (m \wedge p_B) - \frac{1}{2} * (m \wedge \xi^B * d\theta_B) - \right. \\ &\quad \left. - * [m \wedge * (\theta^B \wedge * p_B)] - \frac{1}{2} * (* m \wedge \theta^B) * p_B + \frac{1}{2} * [* (m \wedge \theta^B) \wedge * p_B]\right) + \\ &\quad + R\left(-\theta^B * (m \wedge d\theta_B) - \frac{1}{2} * (m \wedge \xi^B * p_B) + \right. \\ &\quad \left. + * [m \wedge * (\theta^B \wedge * d\theta_B)] + \frac{1}{2} * (* m \wedge \theta^B) * d\theta_B - \frac{1}{2} * [* (m \wedge \theta^B) \wedge * d\theta_B]\right) \end{aligned}$$

where

$$m := M dM' - M' dM. \quad (3.17)$$

The brackets of the boost and rotation constraints and the scalar one:

$$\begin{aligned} \{B(a), S(M)\} &= -B\left(M[\theta^B * (p_B \wedge a) - \frac{1}{2} a * (p_B \wedge \theta^B) + d\xi_B * (a \wedge * \theta^B)]\right) + \\ &\quad + R\left(* (dM \wedge a)\right), \end{aligned} \quad (3.18)$$

$$\begin{aligned} \{R(b), S(M)\} &= -R\left(M[\theta^B * (p_B \wedge b) - \frac{1}{2} b * (p_B \wedge \theta^B) + d\xi_A * (b \wedge * \theta^A)]\right) - \\ &\quad - B\left(* (dM \wedge b)\right). \end{aligned} \quad (3.19)$$

The brackets of the vector constrains:

$$\begin{aligned}
\{V(\vec{M}), V(\vec{M}')\} &= V(\mathcal{L}_{\vec{M}}\vec{M}') \equiv V([\vec{M}, \vec{M}']), \\
\{V(\vec{M}), S(M)\} &= S(\mathcal{L}_{\vec{M}}M), \\
\{V(\vec{M}), B(a)\} &= B(\mathcal{L}_{\vec{M}}a), \\
\{V(\vec{M}), R(b)\} &= R(\mathcal{L}_{\vec{M}}b),
\end{aligned} \tag{3.20}$$

where  $\mathcal{L}_{\vec{M}}$  denotes the Lie derivative on  $\Sigma$  with respect to the vector field  $\vec{M}$ .

Thus the Poisson bracket of any pair of the constraints  $S(M)$ ,  $V(\vec{M})$ ,  $B(a)$  and  $R(b)$  is a combination of the constraints. Since the Hamiltonian (3.15) is a sum of the constraints each of the constraints listed above is preserved by the time evolution hence the list of the constraints is *complete*. All these mean that the constraints are of the *first class*. Note, however, that the constraint algebra is not a Lie algebra—most of the Poisson brackets are combinations of the constraints smeared with fields being functions of the canonical variables.

### 3.3 Discussion

The main conclusion is that the Legendre transformation applied to the action (1.1) as a functional of cotetrad fields leads to a well defined ADM-like Hamiltonian formulation of TEGR. It is a constrained Hamiltonian system with first class constraints only. As a consequence of parameterizing the “timelike” part  $\theta_{\perp}^A$  of the cotetrad by means of the lapse  $N$  and the shift  $\vec{N}$  (see Equation (2.6)) there appear in this formulation the scalar and the vector constraints.

#### 3.3.1 Gauge transformations

Transformations of the canonical variables generated by a first class constraint are usually interpreted as gauge transformations. In the case of this Hamiltonian system it is easy to find a clear interpretation of transformations generated by the vector constraint  $V(\vec{M})$ , the boost constraint  $B(a)$  and the rotation constraint  $R(b)$ . As usual, gauge transformations given by the vector constraint  $V(\vec{M})$  coincides with pull-back of the canonical variables generated by diffeomorphisms moving points along integral curves of the vector field  $\vec{M}$  (see (5.121)).

Regarding the constraints  $B(a)$  and  $R(b)$  we note first that the Poisson brackets (3.16) are related closely to the Lie brackets of the Lie algebra of the Lorentz group. Indeed, there exists a basis  $(\beta^i, \rho^j)$  ( $i, j = 1, 2, 3$ ) of the Lie algebra consisting of generators of boosts  $(\beta^i)$  and of generators of rotations  $(\rho^j)$  such that

$$[\beta^i, \beta^j] = -\tilde{\epsilon}^{ijk} \rho^k, \quad [\rho^i, \rho^j] = \tilde{\epsilon}^{ijk} \beta^k, \quad [\beta^i, \rho^j] = \tilde{\epsilon}^{ijk} \beta^k,$$

where  $\tilde{\epsilon}^{ijk}$  is an antisymmetric symbol such that  $\tilde{\epsilon}^{123} = 1$ . Defining

$$\beta(A) := \beta^i A_i, \quad \rho(B) := \rho^i B_i, \quad [\tilde{*}(A \wedge B)]_l := \tilde{\epsilon}^{ijk} A_i B_j \delta_{kl}$$

we can rewrite the Lie brackets above in the following form

$$\begin{aligned} [\beta(A), \beta(A')] &= -\rho(\tilde{*}(A \wedge A')), \\ [\rho(B), \rho(B')] &= \rho(\tilde{*}(B \wedge B')), \\ [\beta(A), \rho(B)] &= \beta(\tilde{*}(A \wedge B)) \end{aligned} \tag{3.21}$$

Taking into account that for any one-forms  $a, b$  on  $\Sigma$  the  $l$ -th component

$$[\tilde{*}(a \wedge b)]_l = \epsilon^{ijk} a_i b_j q_{kl}$$

the close relations between (3.16) and (3.21) becomes evident and we are allowed to conclude that the constraints  $B(a)$  and  $R(b)$  generate local Lorentz transformations of the canonical variables— $B(a)$  generates local boosts and  $R(b)$  local rotations.

This conclusion can be also justified directly by proving that the transformations generated by  $B(a)$  and  $R(b)$  preserve the spacetime metric  $g$  given by (2.2). By virtue of (2.12) it is enough to show that the transformations preserve the metric  $q$  given by (2.9). Any such a transformation maps  $\theta^A$  to  $\theta^A(\lambda)$  being a value of a solution  $\lambda \mapsto \theta^A(\lambda)$  of one of the following differential equations

$$\begin{aligned} \frac{d\theta_i^A}{d\lambda} &= \{\theta_i^A, B(a(\lambda))\} = \left( \frac{\delta B(a(\lambda))}{\delta p_A} \right)_i = a_i(\lambda) \xi^A, \\ \frac{d\theta_i^A}{d\lambda} &= \{\theta_i^A, R(b(\lambda))\} = \left( \frac{\delta R(b(\lambda))}{\delta p_A} \right)_i = \epsilon^{jk}_i b_j(\lambda) \theta_k^A \end{aligned} \tag{3.22}$$

(see (5.19) and (5.21)) with the initial condition  $\theta_i^A(\lambda = 0) = \theta_i^A$ . Using (3.22) and (2.10) one can easily show that for the resulting transformations  $q \mapsto q(\lambda)$

$$\frac{dq_{ij}}{d\lambda} = 0.$$

Note finally that it follows directly from the r.h.s. of the last equation in (3.22) that the constraint  $R(b)$  generates rotations of  $\theta^A$ .

### 3.3.2 Hamiltonian formulation of TEGR versus a simple model described in [17]

The action (1.1) can be alternatively expressed as [14]

$$S[\theta^A] = -\frac{1}{2} \int d\theta^A \wedge \star d\theta_A - (\star d \star \theta^A) \wedge d \star \theta_A - \frac{1}{2} (d\theta^A \wedge \theta_A) \wedge \star (d\theta^B \wedge \theta_B). \tag{3.23}$$

Omitting the last two terms at the r.h.s. of this expression we obtain the action

$$s[\theta^A] = -\frac{1}{2} \int d\theta^A \wedge \star d\theta_A \quad (3.24)$$

defining the dynamics of a theory considered in [17]. The phase space of that theory coincides with the phase space of TEGR described in this paper. The Legendre transformation defined by (3.24) turns out to be invertible (there are no primary constraints) and one obtains the following Hamiltonian:

$$h[\theta^A, p_A, N, \vec{N}] = s(N) + v(\vec{N}), \quad (3.25)$$

where

$$s(N) = \int_{\Sigma} N \left( \frac{1}{2} p^A \wedge \star p_A - \xi^A dp_A + \frac{1}{2} d\theta^A \wedge \star d\theta_A \right)$$

is a smeared scalar constraint and  $v(\vec{N}) \equiv V(\vec{M})$  is a smeared vector constraint. These secondary constraints are the only constraints and they are of the first class:

$$\begin{aligned} \{s(M), s(M')\} &= v(\vec{m}), \\ \{v(\vec{M}), s(M)\} &= s(\mathcal{L}_{\vec{M}} M), \\ \{v(\vec{M}), v(\vec{M}')\} &= v([\vec{M}, \vec{M}']), \end{aligned}$$

where  $m$  is given by (3.17). This means, in particular, that in this model there are no gauge transformations which could be interpreted as local Lorentz transformations.

Taking this simple theory as a reference point we see that the last two terms at the r.h.s. of (3.23) are responsible for the following features of this formulation of TEGR:

1. the non-invertibility of the Legendre transformation (3.3) hence
2. the presence of the primary constraints  $B(a)$  and  $R(b)$  hence
3. the existence of gauge transformations interpreted as local Lorentz transformations (this fact can be also seen at the level of the Lagrangian formulation of TEGR [14]);
4. the more complicated form of the scalar constraint  $S(N)$  hence
5. the more complicated form of the Poisson bracket of the scalar constraints.

### 3.3.3 Structure of the scalar constraints

Let us comment on the structure of the scalar constraints  $S(M)$  of TEGR and  $s(M)$  of the simple theory<sup>2</sup>. Let  $\alpha = \alpha^A \otimes v_A$  and  $\beta = \beta^B \otimes v_B$  be two-forms on  $\mathcal{M}$  valued in  $\mathbb{M}$ .

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<sup>2</sup>Description of the properties of  $s(M)$  presented below comes from [17].

Given coreper  $(\theta^A)$  on the manifold, which via the metric  $g$  defines the Hodge operator  $\star$ , one can introduce bilinear maps

$$\begin{aligned}(\alpha, \beta) &\mapsto \mathbf{K}(\alpha, \beta) := \frac{1}{2}(\alpha^A \wedge \theta_B) \wedge \star(\beta^B \wedge \theta_A) - \frac{1}{4}(\alpha^A \wedge \theta_A) \wedge \star(\beta^B \wedge \theta_B), \\(\alpha, \beta) &\mapsto \mathbf{k}(\alpha, \beta) := \frac{1}{2}\alpha^A \wedge \star\beta_A.\end{aligned}$$

Similarly, let  $\alpha = \alpha^A \otimes v_A$  and  $\beta = \beta^B \otimes v_B$  be two-forms on  $\Sigma$  valued in  $\mathbb{M}$ . Given restricted coreper  $(\theta^A \equiv \underline{\theta}^A)$  on the manifold, which via the metric  $q$  defines the Hodge operator  $*$ , one can introduce bilinear maps

$$\begin{aligned}(\alpha, \beta) &\mapsto K(\alpha, \beta) := \frac{1}{2}(\alpha^A \wedge \theta_B) \wedge *(\beta^B \wedge \theta_A) - \frac{1}{4}(\alpha^A \wedge \theta_A) \wedge *(\beta^B \wedge \theta_B), \\(\alpha, \beta) &\mapsto k(\alpha, \beta) := \frac{1}{2}\alpha^A \wedge *\beta_A.\end{aligned}$$

Note now that the actions (1.1) of TEGR and (3.24) of the simple theory can be written respectively as

$$S[\theta^A] = - \int \mathbf{K}(d\theta, d\theta), \quad s[\theta^A] = - \int \mathbf{k}(d\theta, d\theta),$$

where  $d\theta = d\theta^A \otimes v_A$ . On the other hand the scalar constraints  $S(M)$  of TEGR and  $s(M)$  of the simple theory can be expressed as

$$S(M) = \int K(p, p) - \xi^A dp_A + K(d\theta, d\theta), \quad s(M) = \int k(p, p) - \xi^A dp_A + k(d\theta, d\theta)$$

where  $d\theta = d\theta^B \otimes v_B$ .

We see thus that the form of each scalar constraint is closely related to the form of the corresponding action. Moreover, the relations in both cases of TEGR and the simple theory follow the same pattern.

### 3.3.4 Comparison with the Hamiltonian framework presented in [12]

A complete analysis of a Hamiltonian framework of TEGR considered as a theory of a cotetrad field was presented in [12]<sup>3</sup>. The main difference between the approach of [12] and that presented in this paper consists in the different way of parameterizing the non-dynamical part of the configurational degrees of freedom: in [12] it is parameterized naturally by  $\theta_\perp^A$ , here we use the lapse  $N$  and the shift  $\vec{N}$  (see Equation (2.6)). Moreover,

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<sup>3</sup>More precisely, the authors of [12] consider TEGR with the unimodular condition imposed but it is easy to read off from their results the Hamiltonian formulation of the standard TEGR.

in [12] an other action than (1.1) is used as a start point of the analysis. Consequently, the resulting Hamiltonian, the set of constraints and the constraint algebra differ significantly from those derived in this paper. In particular, the constraint algebra presented in [12] is much simpler and is in fact a Lie algebra.

## 4 Derivation of the Hamiltonian

Let us recall that to describe the canonical framework of TEGR we use a Hamiltonian formalism adapted to differential forms [16, 5] (see also [17]).

### 4.1 3 + 1 decomposition of the action

It was shown in [17] that if  $\alpha, \beta$  are  $k$ -forms on  $\mathcal{M}$  and  $\star$  is the Hodge operator given by the spacetime metric  $g$  (defined by Equation (2.2)) then

$$\alpha \wedge \star \beta = -N^{-1} \mathbf{d}t \wedge (\alpha_{\perp} - \vec{N} \lrcorner \alpha) \wedge \star (\beta_{\perp} - \vec{N} \lrcorner \beta) + N \mathbf{d}t \wedge \underline{\alpha} \wedge \star \underline{\beta}, \quad (4.1)$$

where  $\star$  is the Hodge operator given by the Euclidean metric  $q$  (defined by Equation (2.9)) on  $\Sigma_t$ , and  $N$  and  $\vec{N}$  are, respectively, the lapse and the shift appearing in (2.6)<sup>4</sup>.

To obtain a 3 + 1 decomposition of the action (1.1) we apply the decomposition (4.1) separately to the first and the second terms under the integral at the r.h.s. of (1.1). By virtue of (2.4)

$$(d\theta^A \wedge \theta_B)_{\perp} = (d\theta^A)_{\perp} \wedge \underline{\theta}_B + \underline{d\theta}^A \wedge (\theta_B)_{\perp} = \mathcal{L}_{\partial_t} \underline{\theta}^A \wedge \underline{\theta}_B - d(\theta_{\perp}^A) \wedge \underline{\theta}_B + \underline{d\theta}^A \wedge (\theta_B)_{\perp}.$$

and

$$\underline{d\theta}^A \wedge \underline{\theta}_B = d\underline{\theta}^A \wedge \underline{\theta}_B.$$

In order to make further calculations more transparent we introduce the following abbreviations:

$$\begin{aligned} F^A{}_B &\equiv d\underline{\theta}^A \wedge \underline{\theta}_B, \\ E^A{}_B &\equiv -d(\theta_{\perp}^A) \wedge \underline{\theta}_B + \underline{d\theta}^A \wedge \theta_{B\perp} - \vec{N} \lrcorner F^A{}_B. \end{aligned} \quad (4.2)$$

Since now we will moreover apply the simplified notation (3.1) and (3.2). Now we can write

$$(d\theta^A \wedge \theta_B)_{\perp} - \vec{N} \lrcorner \underline{d\theta}^A \wedge \underline{\theta}_B = \dot{\theta}^A \wedge \theta_B + E^A{}_B.$$

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<sup>4</sup>In fact, to prove (4.1) it is not necessary to assume that the spacetime metric  $g$  is defined by a cotetrad—it is sufficient to assume (2.12).

At this point we can easily decompose the action (1.1) obtaining thereby

$$S[\theta^A, N, \vec{N}] = \int \frac{1}{2N} dt \wedge (\dot{\theta}^A \wedge \theta_B + E^A_B) \wedge *(\dot{\theta}^B \wedge \theta_A + E^B_A) - \frac{N}{2} dt \wedge F^A_B \wedge *F^B_A - \\ - \frac{1}{4N} dt \wedge (\dot{\theta}^A \wedge \theta_A + E^A_A) \wedge *(\dot{\theta}^B \wedge \theta_B + E^B_B) + \frac{N}{4} dt \wedge F^A_A \wedge *F^B_B. \quad (4.3)$$

## 4.2 Legendre transformation

Note that in the decomposed action (4.3) there is no Lie derivative of  $N$  and  $\vec{N}$  with respect to  $\partial_t$ . Therefore since now we will treat the lapse and the shift as Lagrange multipliers. Consequently, the only dynamical variables are the one-forms  $\theta^A$ . Thus a point in the phase space of the theory is a collection  $(\theta^A, p_B)$  ( $A, B = 0, 1, 2, 3$ ), where  $(\theta^A)$  are one-forms on  $\Sigma$  such that in each point  $x \in \Sigma$  the components  $(\theta^A_i(x))$  form a matrix of the maximal rank, and the momentum  $p_A$  conjugate to  $\theta^A$  is a two-form on  $\Sigma$ .

Let us recall that we denoted by  $L$  the four-form on  $\mathcal{M}$  being the integrand in (1.1). The Legendre transformation reads

$$p_A = \frac{\partial L_\perp}{\partial \dot{\theta}^A} = N^{-1} \left( \theta_B \wedge *(\dot{\theta}^B \wedge \theta_A + E^B_A) - \frac{1}{2} \theta_A \wedge *(\dot{\theta}^B \wedge \theta_B + E^B_B) \right). \quad (4.4)$$

and allow us to introduce a Hamiltonian

$$H_0[\theta^A, \dot{\theta}^A, N, \vec{N}] := \int_\Sigma \dot{\theta}^A \wedge p_A - L_\perp = \int_\Sigma \frac{1}{2N} (\dot{\theta}^A \wedge \theta_B - E^A_B) \wedge *(\dot{\theta}^B \wedge \theta_A + E^B_A) - \\ - \frac{1}{4N} (\dot{\theta}^A \wedge \theta_A - E^A_A) \wedge *(\dot{\theta}^B \wedge \theta_B + E^B_B) + \frac{N}{2} F^A_B \wedge *F^B_A - \frac{N}{4} F^A_A \wedge *F^B_B.$$

expressed as a functional depending on  $\theta^A$ , Lie derivatives  $\dot{\theta}^A$ , the lapse and the shift. Our goal now is to express the Hamiltonian as a functional of  $\theta^A$ , the momentum  $p_A$ , the lapse and the shift.

Let us first gather the terms containing  $\dot{\theta}^A$ :

$$H_0 = \int_\Sigma \frac{1}{2N} \left( (\dot{\theta}^A \wedge \theta_B) \wedge *(\dot{\theta}^B \wedge \theta_A) - \frac{1}{2} (\dot{\theta}^A \wedge \theta_A) \wedge *(\dot{\theta}^B \wedge \theta_B) \right) - \\ - \frac{1}{2N} E^A_B \wedge *E^B_A + \frac{1}{4N} E^A_A \wedge *E^B_B + \frac{N}{2} F^A_B \wedge *F^B_A - \frac{N}{4} F^A_A \wedge *F^B_B. \quad (4.5)$$

For any one-forms  $\alpha^A, \beta^B$  on  $\Sigma$  the following map

$$(\alpha^A, \beta^B) \mapsto G(\alpha^A, \beta^B) := \left( (\alpha^A \wedge \theta_B) \wedge *(\beta^B \wedge \theta_A) - \frac{1}{2} (\alpha^A \wedge \theta_A) \wedge *(\beta^B \wedge \theta_B) \right) = \\ = (q^{ik} q^{jl} - q^{ij} q^{kl}) \theta_{A(i} \alpha_{j)}^A \theta_{B(k} \beta_{l)}^B \in \quad (4.6)$$



is bilinear and symmetric. Thus

$$\begin{aligned} \left( (\dot{\theta}^A \wedge \theta_B) \wedge *(\dot{\theta}^B \wedge \theta_A) - \frac{1}{2}(\dot{\theta}^A \wedge \theta_A) \wedge *(\dot{\theta}^B \wedge \theta_B) \right) &= G(\dot{\theta}^A, \dot{\theta}^B) = \\ &= (q^{ik}q^{jl} - q^{ij}q^{kl})\theta_{A(i}\dot{\theta}_{j)}^A\theta_{B(k}\dot{\theta}_{l)}^B \epsilon. \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} H_0 = \int_{\Sigma} \frac{1}{2N} \left( G(\dot{\theta}^A, \dot{\theta}^B) - E^A{}_B \wedge *E^B{}_A + \frac{1}{2}E^A{}_A \wedge *E^B{}_B \right) + \\ + \frac{N}{2}F^A{}_B \wedge *F^B{}_A - \frac{N}{4}F^A{}_A \wedge *F^B{}_B. \end{aligned} \quad (4.8)$$

To replace the “velocity”  $\dot{\theta}^A$  by the momentum  $p_A$  in the Hamiltonian  $H_0$  let us try to invert the Legendre transformation (4.4). We start by acting on both sides of (4.4) by the Hodge operator  $*$ . For any one-form  $\alpha$  and any  $k$ -form  $\beta$  [17]

$$*(*\beta \wedge \alpha) = \vec{\alpha} \lrcorner \beta, \quad (4.9)$$

hence

$$N*p_A + \vec{\theta}_{B\lrcorner}E^B{}_A - \frac{1}{2}\vec{\theta}_{A\lrcorner}E^B{}_B = -\vec{\theta}_{B\lrcorner}(\dot{\theta}^B \wedge \theta_A) + \frac{1}{2}\vec{\theta}_{A\lrcorner}(\dot{\theta}^B \wedge \theta_B)$$

Denoting

$$\pi_A \equiv N*p_A + \vec{\theta}_{B\lrcorner}E^B{}_A - \frac{1}{2}\vec{\theta}_{A\lrcorner}E^B{}_B \quad (4.10)$$

we obtain

$$\pi_{Aj} = \theta_{B(i}\dot{\theta}_{j)}^B\theta_A^i - \theta_B^i\dot{\theta}_i^B\theta_{Aj} = \theta_{B(i}\dot{\theta}_{j)}^B\theta_A^i - \theta_{B(i}\dot{\theta}_k^Bq^{kj}\theta_{Aj)}. \quad (4.11)$$

Note that by virtue of (2.9)

$$\theta_i^A\theta_A^j = \theta_i^A\theta_{Ak}q^{kj} = g_{ik}q^{kj} = \delta^j_i. \quad (4.12)$$

Using this identity we obtain from (4.11) the following equation

$$\pi_{Aj}\theta_k^A = \theta_{B(j}\dot{\theta}_k^B - \theta_B^i\dot{\theta}_i^Bq_{jk}.$$

Contracting both sides of the last formula with  $q^{jk}$  we get

$$\pi_{Aj}\theta^{Aj} = -2\theta_B^i\dot{\theta}_i^B.$$

Thus

$$\pi_{Aj}\theta_k^A - \frac{1}{2}\pi_{Ai}\theta^{Ai}q_{jk} = \theta_{A(j}\dot{\theta}_k^A. \quad (4.13)$$

It is evident now that the Legendre transformation is not invertible. The source of the non-invertibility is twofold:

1. treating  $\dot{\theta}_i^A$  of a fixed  $i$  as a four-component vector we see that in the expression (4.13) there appear only contractions of  $\dot{\theta}_i^A$  with the three linearly independent vectors  $\{\theta_j^A\}$  ( $j = 1, 2, 3$ ) while the contraction  $\xi_A \dot{\theta}_i^A$  is missing (recall that at each point of  $\Sigma$  the values of functions  $(\xi^A, \theta_i^A)$  form a basis of  $\mathbb{M}$ ).
2. only the symmetric part of the tensor  $\theta_{Ai} \dot{\theta}_j^A$  appears in the expression.

This means that information encoded in  $\dot{\theta}^A$  is reduced by the transformation. To analyze the reduction let us fix a point  $x \in \Sigma$  and values of  $\theta^A$ ,  $d\theta^A$ ,  $N$  and  $\vec{N}$  at this point and treat (4.4) as a map transforming  $\dot{\theta}^A(x)$  to  $p_A(x)$ . This map can be seen as a composition  $I \circ P$  of a injection  $I$  and a projection  $P$ . Indeed,  $P$  is a map which maps 12 independent quantities  $\dot{\theta}_i^A(x)$  to  $\theta_{A(j}(x) \dot{\theta}_{k)}^A(x)$  loosing information encoded in 6 quantities  $\xi_A(x) \dot{\theta}_i^A(x)$  and  $\theta_{A[j}(x) \dot{\theta}_{k]}^A(x)$ . It follows from Equations (4.11) and (4.10) that the value  $\theta_{A(j}(x) \dot{\theta}_{k)}^A(x)$  unambiguously gives the value  $p_{ij}^A(x)$  and this mapping is what we called  $I$  above. On the other hand we see from Equation (4.13) that once we have  $p_{ij}^A(x)$  we have also  $\theta_{A(j}(x) \dot{\theta}_{k)}^A(x)$  which means that  $I$  is a injection. Hence the image of the map  $I \circ P : \dot{\theta}^A(x) \mapsto p_A(x)$  is 6-dimensional. Therefore there should be 6 independent constraints imposed on 12 quantities  $p_{Aij}(x)$ :

1. contracting  $\xi^A$  with  $\pi_{Aj}$  given by (4.11) we see that by virtue of (2.10)

$$0 = \pi_{Ai} \xi^A$$

These three constraints can be equivalently expressed as

$$\pi_A \xi^A = -\frac{1}{3!} \pi_A \epsilon^A{}_{BCD} *(\theta^B \wedge \theta^C \wedge \theta^D) = 0. \quad (4.14)$$

2. extracting the antisymmetric part of both sides of (4.13) we obtain the three remaining constraints

$$\pi_{A[j} \theta_{k]}^A = 0$$

or equivalently

$$\pi_A \wedge \theta^A = 0. \quad (4.15)$$

Note that the conditions (4.14) and (4.15) contain the one-form  $\pi_A$  which depends on the laps  $N$  and the shift  $\vec{N}$ . Therefore it is not obvious that the conditions define *constraints on the phase space*. We will show in the next subsection that the laps and the shift can be removed from the conditions which leave expressions depending on  $\theta^A$ ,  $d\theta^A$  and  $p_A$  only. In other words we will show there that (4.14) and (4.15) are in fact *primary constraints*. Moreover, we will prove there that they are 6 independent restrictions on the values of  $p_A$  at every point of  $\Sigma$ .

The non-invertibility of the Legendre transformation is not an obstacle for expressing the Hamiltonian  $H_0$  as a functional of the momentum  $p_A$ —note that the Hamiltonian depends on the “velocity”  $\dot{\theta}_i^A$  via the combination  $\theta_{A(i}\dot{\theta}_{j)}^A$  (see the formula (4.7)) and according to Equation (4.13) this combination is a function of the momentum. The latter equation can be rewritten as

$$\theta_{A(j}\dot{\theta}_{k)}^A = \theta_{A(j}[\pi_k^A - \frac{1}{2}(\tilde{\theta}^C \lrcorner \pi_C)\theta_k^A].$$

and consequently the term (4.7) can be expressed in the following form

$$\begin{aligned} G(\dot{\theta}^A, \dot{\theta}^B) &= (q^{ik}q^{jl} - q^{ij}q^{kl})\theta_{A(i}\dot{\theta}_{j)}^A\theta_{B(k}\dot{\theta}_{l)}^B \epsilon = \\ &= (q^{ik}q^{jl} - q^{ij}q^{kl})\left(\theta_{A(i}[\pi_j^A - \frac{1}{2}(\tilde{\theta}^C \lrcorner \pi_C)\theta_j^A]\right)\left(\theta_{B(j}[\pi_k^B - \frac{1}{2}(\tilde{\theta}^D \lrcorner \pi_D)\theta_k^B]\right)\epsilon = \\ &= G(\pi^A - \frac{1}{2}(\tilde{\theta}^C \lrcorner \pi_C)\theta^A, \pi^B - \frac{1}{2}(\tilde{\theta}^D \lrcorner \pi_D)\theta^B) \end{aligned}$$

which allows us to rewrite (4.8) as

$$\begin{aligned} H_0[\theta^A, p_A, N, \vec{N}] &= \int_{\Sigma} \frac{1}{2N} \left[ G\left(\pi^A - \frac{1}{2}(\tilde{\theta}^C \lrcorner \pi_C)\theta^A, \pi^B - \frac{1}{2}(\tilde{\theta}^D \lrcorner \pi_D)\theta^B\right) - \right. \\ &\quad \left. - E^A{}_B \wedge *E^B{}_A + \frac{1}{2}E^A{}_A \wedge *E^B{}_B \right] + \frac{N}{2}F^A{}_B \wedge *F^B{}_A - \frac{N}{4}F^A{}_A \wedge *F^B{}_B, \quad (4.16) \end{aligned}$$

where  $\pi_A$  is a function of  $\theta^A, p_B, N, \vec{N}$  given by (4.10).

Let us emphasize that this Hamiltonian is defined not on the whole phase space but only on the image of the Legendre transformation  $(\theta^A, \dot{\theta}^A) \mapsto (\theta^A, p_A)$  given by (4.4), that is, on a subset of the phase space distinguished by vanishing of the primary constraints (4.14) and (4.15).

The Hamiltonian (4.16) can be expressed as an explicit function of the configuration variable  $\theta^A$ , the momentum  $p_A$  and the Lagrangian multipliers  $N$  and  $\vec{N}$ . Before we will do this let us first rewrite the primary constraints (4.14) and (4.15) in a form free of  $N$  and  $\vec{N}$ .

### 4.3 Primary constraints

The goal of this section is to remove the lapse  $N$  and the shift  $\vec{N}$  from the conditions (4.14) and (4.15). Let us start by stating and proving an auxiliary identity.

Let  $\alpha$  be a  $k$ -form on  $\Sigma$ . Then

$$\theta^A \wedge (\tilde{\theta}_{A \lrcorner} \alpha) = k\alpha \quad (4.17)$$

*Proof.* Using (4.12) we calculate

$$\begin{aligned}\theta^A \wedge (\vec{\theta}_A \lrcorner \alpha) &= \theta_i^A dx^i \wedge (\theta_A^j \partial_j \lrcorner \frac{1}{k!} \alpha_{a_1 \dots a_k} dx^{a_1} \wedge \dots \wedge dx^{a_k}) = \\ &= \theta_i^A \theta_A^j \frac{k}{k!} \alpha_{ja_2 \dots a_k} dx^i \wedge dx^{a_2} \wedge \dots \wedge dx^{a_k} = k \frac{1}{k!} \alpha_{ia_2 \dots a_k} dx^i \wedge dx^{a_2} \wedge \dots \wedge dx^{a_k} = k\alpha.\end{aligned}$$

□

It will be convenient to denote

$$\rho_A \equiv N^* p_A + \vec{\theta}_B \lrcorner E^B_A. \quad (4.18)$$

Then

$$\pi_A = \rho_A - \frac{1}{2} \vec{\theta}_A \lrcorner E^B_B. \quad (4.19)$$

Now let us express all the forms above as explicite functions of  $\theta^A, p_B, N, \vec{N}$ . To this end we set into (4.2) the function  $\theta_\perp^A$  written as in (2.6). Then with application of (2.10) and (2.11) we obtain in turn

$$\begin{aligned}E^A_B &= -\xi^A dN \wedge \theta_B - N(d\xi^A \wedge \theta_B - d\theta^A \wedge \xi_B) - \mathcal{L}_{\vec{N}} \theta^A \wedge \theta_B, \\ \vec{\theta}_A \lrcorner E^A_B &= N \left( -(\vec{\theta}_A \lrcorner d\xi^A) \theta_B + d\xi_B + \vec{\theta}_A \lrcorner d\theta^A \xi_B \right) - (\vec{\theta}_A \lrcorner \mathcal{L}_{\vec{N}} \theta^A) \theta_B + \mathcal{L}_{\vec{N}} \theta^A \vec{\theta}_A \lrcorner \theta_B, \\ E^A_A &= -N(d\xi^A \wedge \theta_A - d\theta^A \wedge \xi_A) - \mathcal{L}_{\vec{N}} \theta^A \wedge \theta_A, \\ \rho_A &= N \left( *p_A - (\vec{\theta}_C \lrcorner d\xi^C) \theta_A + d\xi_A + \vec{\theta}_C \lrcorner d\theta^C \xi_A \right) - (\vec{\theta}_C \lrcorner \mathcal{L}_{\vec{N}} \theta^C) \theta_A + \mathcal{L}_{\vec{N}} \theta^C \vec{\theta}_C \lrcorner \theta_A,\end{aligned} \quad (4.20)$$

where  $\mathcal{L}_{\vec{N}}$  denotes the Lie derivative on  $\Sigma$  with respect to the shift  $\vec{N}$  (recall that  $\mathcal{L}_{\vec{N}} = d \circ \vec{N} \lrcorner + \vec{N} \lrcorner \circ d$ ).

The condition (4.14) can be simplified as follows

$$0 = \pi_A \xi^A = \rho_A \xi^A = N(\xi^A * p_A - \vec{\theta}_A \lrcorner d\theta^A) = N * (\xi^A p_A + \theta^A \wedge *d\theta_A),$$

where we have used (4.9), (2.10) and (2.11). Consequently, taking into account (2.8) we get

$$\theta^A \wedge *d\theta_A + \xi^A p_A = 0, \quad (4.21)$$

which coincides with (3.5). On the other hand using (4.17), (2.10) and (2.11) we can transform (4.15) as follows

$$\begin{aligned}0 = \theta^A \wedge \pi_A &= \theta^A \wedge \rho_A - E^A_A = N(\theta^A \wedge *p_A + \theta^A \wedge d\xi_A) + \theta_A \wedge \mathcal{L}_{\vec{N}} \theta^A - E^A_A = \\ &= N(\theta^A \wedge *p_A - \xi_A d\theta^A),\end{aligned}$$

hence by virtue of (2.8)

$$\theta^A \wedge *p_A - \xi_A d\theta^A = 0 \quad (4.22)$$

which coincides with (3.6).

Let us fix a point  $x \in \Sigma$  and values of  $\theta^A$  and  $d\theta^A$  at  $x$ . Then 12 quantities  $p_{Aij}(x)$  can be fully encoded in 12 independent quantities

$$\xi^A(x)(*p_A)_i(x), \quad \theta_{[j}^A(x)(*p_A)_{i]}(x), \quad \theta_{(j}^A(x)(*p_A)_{i)}(x).$$

Note now that the conditions (4.21) and (4.22) fix values of the former two quantities (to see this act by  $*$  on both sides of (4.21)). This means that these conditions are *independent*. Since there are 6 of them and since they do not contain the lapse and the shift they are 6 independent *primary constraints* on the phase space.

Recall that in the previous subsection we concluded that, given values of  $\theta^A$ ,  $d\theta^A$ , the lapse and the shift at  $x$ , there are 6 independent constraints imposed on  $p_{Aij}(x)$  being values of the Legendre transformation (4.4). This means that there are no other primary constraints than (4.21) and (4.22).

Let us finally note that setting to (4.4) the two-forms  $E^A_B$  and  $E^B_B$  expressed as in (4.20) we obtain the formula (3.4).

## 4.4 An explicite form of the Hamiltonian

Now we begin a long series of transformations of the Hamiltonian (4.16) aimed at expressing it as an explicite functional of the canonical variables  $\theta^A$  and  $p_A$ , the lapse  $N$  and the shift  $\vec{N}$ . Let us start by introducing and proving some auxiliary formulae which will be repeatedly used in the sequel.

### 4.4.1 Auxiliary formulae

We will often apply an identity linking the contraction  $\vec{\theta}^B \lrcorner \theta^A$  with  $\eta^{AB}$ :

$$\vec{\theta}^B \lrcorner \theta^A = \eta^{AB} + \xi^A \xi^B. \quad (4.23)$$

*Proof.* Using the components of the metric  $g^{-1}$  inverse to  $g$  to raise the space-time indices (here: lower case Greek letters) we obtain from (2.2)

$$\theta_{A\alpha} \theta^{A\beta} = \delta^\beta_\alpha,$$

which means that

$$\theta_{A\alpha} \theta^{B\alpha} = \delta^B_A.$$

Raising the index  $A$  we obtain

$$\theta_\alpha^A \theta_\beta^B g^{\alpha\beta} = \eta^{AB}.$$

Setting to this equation the components of  $g^{-1}$  expressed as [15]

$$g^{00} = -N^{-2}, \quad g^{0i} = N^{-2}N^i, \quad g^{ij} = q^{ij} - N^{-2}N^iN^j$$

we obtain

$$\eta^{AB} = -\frac{1}{N^2}(\theta_{\perp}^A - \vec{N} \lrcorner \theta^A)(\theta_{\perp}^B - \vec{N} \lrcorner \theta^B) + \vec{\theta}^B \lrcorner \theta^A = -\xi^A \xi^B + \vec{\theta}^B \lrcorner \theta^A,$$

where in the last step we applied (2.6).  $\square$

For any one-forms  $\alpha$  and  $\beta$  the following formulae hold:

$$\alpha \wedge * \beta = (\vec{\alpha} \lrcorner \beta) \epsilon, \quad (4.24)$$

$$(\alpha \wedge \theta^B) \wedge *(\beta \wedge \theta^A) = -(\vec{\theta}^A \lrcorner \alpha)(\vec{\theta}^B \lrcorner \beta) \epsilon + (\eta^{AB} + \xi^A \xi^B) \alpha \wedge * \beta. \quad (4.25)$$

*Proof of (4.24).* This formula follows immediately from a definition of the Hodge operator  $*$  (see e.g. [17]).  $\square$

*Proof of (4.25).*

$$\begin{aligned} (\alpha \wedge \theta^B) \wedge *(\beta \wedge \theta^A) &= \alpha \wedge **(\theta^B \wedge *(\beta \wedge \theta^A)) = -\alpha \wedge *[\vec{\theta}^B \lrcorner (\beta \wedge \theta^A)] = \\ &= -\alpha \wedge *[(\vec{\theta}^B \lrcorner \beta) \theta^A] + \alpha \wedge *(\beta \vec{\theta}^B \lrcorner \theta^A) = -(\vec{\theta}^A \lrcorner \alpha)(\vec{\theta}^B \lrcorner \beta) \epsilon + (\eta^{AB} + \xi^A \xi^B) \alpha \wedge * \beta, \end{aligned}$$

where in the second step we used (4.9), and in the last one (4.24) and (4.23).  $\square$

It follows from (4.24) and (4.12) that

$$\theta^A \wedge * \theta_A = (\vec{\theta}^A \lrcorner \theta_A) \epsilon = 3\epsilon. \quad (4.26)$$

Setting in (4.25)  $\alpha = \alpha_A$  and  $\beta = \beta_B$  and assuming summations over  $A$  and  $B$  we get

$$(\alpha_A \wedge \theta^B) \wedge *(\beta_B \wedge \theta^A) = -(\vec{\theta}^A \lrcorner \alpha_A)(\vec{\theta}^B \lrcorner \beta_B) \epsilon + \alpha_A \wedge * \beta^A + (\xi^A \alpha_A) \wedge *(\xi^B \beta_B). \quad (4.27)$$

Similarly, setting in (4.25)  $\alpha = \beta_B$  and  $\beta = \alpha_A$  and assuming summations over  $A$  and  $B$  we obtain

$$(\beta_B \wedge \theta^B) \wedge *(\alpha_A \wedge \theta^A) = -(\vec{\theta}^A \lrcorner \beta_B)(\vec{\theta}^B \lrcorner \alpha_A) \epsilon + \beta_A \wedge * \alpha^A + (\xi^B \beta_B) \wedge *(\xi^A \alpha_A). \quad (4.28)$$

Setting  $\alpha_A = \theta_A$  in (4.27) gives

$$(\theta_A \wedge \theta^B) \wedge *(\beta_B \wedge \theta^A) = -2(\vec{\theta}^B \lrcorner \beta^B) \epsilon = -2\beta_A \wedge * \theta^A \quad (4.29)$$

(the last equality is due to (4.24)). Assume that in the formula just obtained  $\beta_B = \theta_B$ . Applying 4.26 we obtain

$$(\theta_A \wedge \theta^B) \wedge *(\theta_B \wedge \theta^A) = -6\epsilon. \quad (4.30)$$

#### 4.4.2 Calculations

Since now till the end of the paper we will so often apply the formulae (2.10) and (2.11) that it would be troublesome to refer to them each time. Therefore we ask the reader to keep the formulae in mind since they will be used without any reference.

We begin the calculations with the first term of the Hamiltonian (4.16):

$$\begin{aligned}
G\left(\pi^A - \frac{1}{2}(\vec{\theta}^C \lrcorner \pi_C)\theta^A, \pi^B - \frac{1}{2}(\vec{\theta}^D \lrcorner \pi_D)\theta^B\right) &= \\
&= \left(\pi^A - \frac{1}{2}(\vec{\theta}^C \lrcorner \pi_C)\theta^A\right) \wedge \theta_B - \left(\pi^B - \frac{1}{2}(\vec{\theta}^D \lrcorner \pi_D)\theta^B\right) \wedge \theta_A - \\
&\quad - \frac{1}{2}\left(\pi^A - \frac{1}{2}(\vec{\theta}^C \lrcorner \pi_C)\theta^A\right) \wedge \theta_A - \frac{1}{2}\left(\pi^B - \frac{1}{2}(\vec{\theta}^D \lrcorner \pi_D)\theta^B\right) \wedge \theta_B = \\
&= (\pi^A \wedge \theta_B) \wedge *(\pi^B \wedge \theta_A) - \frac{1}{2}(\pi^A \wedge \theta_A) \wedge *(\pi^B \wedge \theta_B) - \\
&\quad - (\vec{\theta}^C \lrcorner \pi_C)\theta^A \wedge \theta_B \wedge *(\pi^B \wedge \theta_A) + \frac{1}{4}(\vec{\theta}^C \lrcorner \pi_C)^2 \theta^A \wedge \theta_B \wedge *(\theta^B \wedge \theta_A).
\end{aligned}$$

By virtue of (4.29) and (4.9)

$$-(\vec{\theta}^C \lrcorner \pi_C)\theta^A \wedge \theta_B \wedge *(\pi^B \wedge \theta_A) = 2(\vec{\theta}^C \lrcorner \pi_C)\pi_A \wedge *\theta^A = 2(\pi_A \wedge *\theta^A) \wedge *(\pi_B \wedge *\theta^B)$$

Applying (4.30) in the first step, (4.9) and (4.24) in the second one we obtain

$$\frac{1}{4}(\vec{\theta}^C \lrcorner \pi_C)^2 \theta^A \wedge \theta_B \wedge *(\theta^B \wedge \theta_A) = -\frac{6}{4}(\vec{\theta}^C \lrcorner \pi_C)^2 \epsilon = -\frac{3}{2}(\pi_A \wedge *\theta^A) \wedge *(\pi_B \wedge *\theta^B).$$

Thus

$$\begin{aligned}
G\left(\pi^A - \frac{1}{2}(\vec{\theta}^C \lrcorner \pi_C)\theta^A, \pi^B - \frac{1}{2}(\vec{\theta}^D \lrcorner \pi_D)\theta^B\right) &= (\pi^A \wedge \theta_B) \wedge *(\pi^B \wedge \theta_A) - \\
&\quad - \frac{1}{2}(\pi^A \wedge \theta_A) \wedge *(\pi^B \wedge \theta_B) + \frac{1}{2}(\pi_A \wedge *\theta^A) \wedge *(\pi_B \wedge *\theta^B).
\end{aligned}$$

Consider now the following map

$$\begin{aligned}
(\alpha^A, \beta^B) &\mapsto \tilde{G}(\alpha^A, \beta^B) := G(\alpha^A, \beta^B) + \frac{1}{2}(\alpha_A \wedge *\theta^A) \wedge *(\beta_B \wedge *\theta^B) \\
&= (q^{ik}q^{jl} - \frac{1}{2}q^{ij}q^{kl})\theta_{A(i}\alpha_{j)}^A \theta_{B(k}\beta_{l)}^B \epsilon \quad (4.31)
\end{aligned}$$

Note that  $\tilde{G}$  is built from (i) the same non-invertible linear mapping  $\alpha_i^A \mapsto \theta_{A(i}\alpha_{j)}^A$  as  $G$  and (ii) the metric  $(q^{ik}q^{jl} - \frac{1}{2}q^{ij}q^{kl})$  related to the metric  $(q^{ik}q^{jl} - q^{ij}q^{kl})$  appearing in (4.6) as follows:

$$(q^{ik}q^{jl} - \frac{1}{2}q^{ij}q^{kl})(q_{kn}q_{lm} - q_{kl}q_{nm}) = \delta_n^i \delta_m^j.$$

Thus

$$G\left(\pi^A - \frac{1}{2}(\vec{\theta}^C \lrcorner \pi_C)\theta^A, \pi^B - \frac{1}{2}(\vec{\theta}^D \lrcorner \pi_D)\theta^B\right) = \tilde{G}(\pi^A, \pi^B).$$

Note now that  $\pi^A$  in  $\tilde{G}(\pi^A, \pi^B)$  undergoes the linear transformation  $\pi_i^A \mapsto \theta_{A(i)}\pi_j^A$ . According to (4.10)  $\pi^A$  contains the term  $-\frac{1}{2}\vec{\theta}^A \lrcorner E^B_B$  which vanishes under the transformation:

$$(\vec{\theta}^A \lrcorner E^B_B)_i = \theta^{Ak} E^B_{Bki} \mapsto \theta_{A(i)} \theta^{Ak} E^B_{B|k|j} = E^B_{B(ij)} = 0.$$

Taking into account Equation (4.19) we see that

$$G\left(\pi^A - \frac{1}{2}(\vec{\theta}^C \lrcorner \pi_C)\theta^A, \pi^B - \frac{1}{2}(\vec{\theta}^D \lrcorner \pi_D)\theta^B\right) = \tilde{G}(\rho^A, \rho^B)$$

and consequently (4.16) can be written as follows:

$$\begin{aligned} H_0[\theta^A, p_B, N, \vec{N}] = \int_{\Sigma} \frac{1}{2N} & \left( \tilde{G}(\rho^A, \rho^B) - E^A_B \wedge *E^B_A + \frac{1}{2} E^A_A \wedge *E^B_B \right) + \\ & + \frac{N}{2} F^A_B \wedge *F^B_A - \frac{N}{4} F^A_A \wedge *F^B_B, \end{aligned} \quad (4.32)$$

Our goal now is to express the terms

$$\begin{aligned} \tilde{G}(\rho^A, \rho^B) - E^A_B \wedge *E^B_A + \frac{1}{2} E^A_A \wedge *E^B_B &= (\rho^A \wedge \theta_B) \wedge *(\rho^B \wedge \theta_A) - \\ - \frac{1}{2} (\rho^A \wedge \theta_A) \wedge *(\rho^B \wedge \theta_B) &+ \frac{1}{2} (\rho_A \wedge * \theta^A) \wedge *(\rho_B \wedge * \theta^B) - E^A_B \wedge *E^B_A + \frac{1}{2} E^A_A \wedge *E^B_B \end{aligned} \quad (4.33)$$

as explicite functions of the canonical variables, the lapse and the shift. To transform the five terms appearing at the r.h.s. of (4.33) we apply Equations (4.20) and obtain in turn: the first term

$$\begin{aligned} (\rho_A \wedge \theta^B) \wedge *(\rho_B \wedge \theta^A) &= \\ &= \left( N[*p_A - (\vec{\theta}_C \lrcorner d\xi^C)\theta_A + d\xi_A] - (\vec{\theta}_C \lrcorner \mathcal{L}_{\vec{N}}\theta^C)\theta_A + \mathcal{L}_{\vec{N}}\theta^C \vec{\theta}_C \lrcorner \theta_A \right) \wedge \theta^B \wedge \\ &\wedge * \left[ \left( N[*p_B - (\vec{\theta}_D \lrcorner d\xi^D)\theta_B + d\xi_B] - (\vec{\theta}_D \lrcorner \mathcal{L}_{\vec{N}}\theta^D)\theta_B + \mathcal{L}_{\vec{N}}\theta^D \vec{\theta}_D \lrcorner \theta_B \right) \wedge \theta^A \right] = \\ &= N^2 \left( *p_A \wedge \theta^B \wedge *(*p_B \wedge \theta^A) + 2(\vec{\theta}^A \lrcorner *p_A)(\vec{\theta}^B \lrcorner d\xi_B)\epsilon + 2p_A \wedge d\xi^A - 2(\vec{\theta}^B \lrcorner d\xi_B)^2\epsilon + \right. \\ &+ d\xi_A \wedge \theta^B \wedge *(d\xi_B \wedge \theta^A) \Big) + N \left( 4(\vec{\theta}_A \lrcorner \mathcal{L}_{\vec{N}}\theta^A)(\vec{\theta}^B \lrcorner *p_B)\epsilon + 2*p_A \wedge \theta^B \wedge *(\mathcal{L}_{\vec{N}}\theta_B \wedge \theta^A) - \right. \\ &- 6(\vec{\theta}_A \lrcorner d\xi^A)(\vec{\theta}_B \lrcorner \mathcal{L}_{\vec{N}}\theta^B)\epsilon + 2d\xi_A \wedge *\mathcal{L}_{\vec{N}}\theta^A \Big) - 2(\vec{\theta}_A \lrcorner \mathcal{L}_{\vec{N}}\theta^A)^2\epsilon + \mathcal{L}_{\vec{N}}\theta_A \wedge \theta^B \wedge *(\mathcal{L}_{\vec{N}}\theta_B \wedge \theta^A), \end{aligned}$$



the second one

$$\begin{aligned}
& -\frac{1}{2}(\rho^A \wedge \theta_A) \wedge *(\rho^B \wedge \theta_B) = -\frac{1}{2}\left(N[*p_A \wedge \theta^A + d\xi_A \wedge \theta^A] + \mathcal{L}_{\vec{N}}\theta^A \wedge \theta_A\right) \wedge \\
& \quad \wedge * \left(N[*p_B \wedge \theta^B + d\xi_B \wedge \theta^B] + \mathcal{L}_{\vec{N}}\theta^B \wedge \theta_B\right) = \\
& = N^2\left(-\frac{1}{2}*p_A \wedge \theta^A \wedge *(*p_B \wedge \theta^B) - *p_A \wedge \theta^A \wedge *(d\xi_B \wedge \theta^B) - \frac{1}{2}(d\xi_A \wedge \theta^A) \wedge *(d\xi_B \wedge \theta^B)\right) + \\
& + N\left(-*p_A \wedge \theta^A \wedge *(\mathcal{L}_{\vec{N}}\theta_B \wedge \theta^B) - (d\xi_A \wedge \theta^A) \wedge *(\mathcal{L}_{\vec{N}}\theta_B \wedge \theta^B)\right) - \frac{1}{2}\mathcal{L}_{\vec{N}}\theta_A \wedge \theta^A \wedge *(\mathcal{L}_{\vec{N}}\theta_B \wedge \theta^B),
\end{aligned}$$

the third one

$$\begin{aligned}
& \frac{1}{2}(\rho_A \wedge *\theta^A) \wedge *(\rho_B \wedge *\theta^B) = \frac{1}{2}\left(N[*p_A \wedge *\theta^A - 2(\vec{\theta}_C \lrcorner d\xi^C)\epsilon] - 2(\vec{\theta}_C \lrcorner \mathcal{L}_{\vec{N}}\theta^C)\epsilon\right) \wedge \\
& \quad \wedge * \left(N[*p_B \wedge *\theta^B - 2(\vec{\theta}_D \lrcorner d\xi^D)\epsilon] - 2(\vec{\theta}_D \lrcorner \mathcal{L}_{\vec{N}}\theta^D)\epsilon\right) = \\
& = N^2\left(\frac{1}{2}p_A \wedge \theta^A \wedge *(p_B \wedge \theta^B) - 2(\vec{\theta}^A \lrcorner *p_A)(\vec{\theta}_C \lrcorner d\xi^C)\epsilon + 2(\vec{\theta}_C \lrcorner d\xi^C)^2\epsilon\right) + \\
& + N\left(-2(\vec{\theta}^A \lrcorner *p_A)(\vec{\theta}_C \lrcorner \mathcal{L}_{\vec{N}}\theta^C)\epsilon + 4(\vec{\theta}_C \lrcorner d\xi^C)(\vec{\theta}_D \lrcorner \mathcal{L}_{\vec{N}}\theta^D)\epsilon\right) + 2(\vec{\theta}_C \lrcorner \mathcal{L}_{\vec{N}}\theta^C)^2\epsilon,
\end{aligned}$$

the fourth one

$$\begin{aligned}
& -E^A{}_B \wedge *E^B{}_A = -\left(-\xi^A dN \wedge \theta_B - N(d\xi^A \wedge \theta_B - d\theta^A \wedge \xi_B) - \mathcal{L}_{\vec{N}}\theta^A \wedge \theta_B\right) \wedge \\
& \wedge * \left(-\xi^B dN \wedge \theta_A - N(d\xi^B \wedge \theta_A - d\theta^B \wedge \xi_A) - \mathcal{L}_{\vec{N}}\theta^B \wedge \theta_A\right) = -2NdN \wedge \theta_B \wedge *(d\theta^B) - \\
& \quad N^2\left(-d\xi^A \wedge \theta_B \wedge *(d\xi^B \wedge \theta_A) - (d\theta^A \wedge \xi_B) \wedge *(d\theta^B \wedge \xi_A)\right) + \\
& + N\left(-2d\xi^A \wedge \theta_B \wedge *(\mathcal{L}_{\vec{N}}\theta^B \wedge \theta_A) + 2d\theta^A \wedge \xi_B \wedge *(\mathcal{L}_{\vec{N}}\theta^B \wedge \theta_A)\right) - \mathcal{L}_{\vec{N}}\theta^A \wedge \theta_B \wedge *(\mathcal{L}_{\vec{N}}\theta^B \wedge \theta_A)
\end{aligned} \tag{4.34}$$

and, finally, the fifth one

$$\begin{aligned}
& \frac{1}{2}E^A{}_A \wedge *E^B{}_B = \frac{1}{2}\left(-N(d\xi^A \wedge \theta_A - d\theta^A \wedge \xi_A) - \mathcal{L}_{\vec{N}}\theta^A \wedge \theta_A\right) \wedge \\
& \quad \wedge * \left(-N(d\xi^B \wedge \theta_B - d\theta^B \wedge \xi_B) - \mathcal{L}_{\vec{N}}\theta^B \wedge \theta_B\right) = \\
& = N^2\left(\frac{1}{2}d\xi^A \wedge \theta_A \wedge *(d\xi^B \wedge \theta_B) - d\xi^A \wedge \theta_A \wedge *(d\theta^B \wedge \xi_B) + \frac{1}{2}d\theta^A \wedge \xi_A \wedge *(d\theta^B \wedge \xi_B)\right) + \\
& + N\left(d\xi^A \wedge \theta_A \wedge *(\mathcal{L}_{\vec{N}}\theta^B \wedge \theta_B) - d\theta^A \wedge \xi_A \wedge *(\mathcal{L}_{\vec{N}}\theta^B \wedge \theta_B)\right) + \frac{1}{2}\mathcal{L}_{\vec{N}}\theta^A \wedge \theta_A \wedge *(\mathcal{L}_{\vec{N}}\theta^B \wedge \theta_B).
\end{aligned}$$

Note now that each of the five terms consists of terms proportional to  $N^2$ ,  $N$  and ones which do not depend on  $N$ . Moreover, in (4.34) there is one term proportional to

$NdN$ . Let us now gather the corresponding terms obtaining thereby a decomposition of (4.33) into terms proportional to  $N^2, N, NdN$  and that independent of  $N$ .

Gathering the terms we will try to simplify the formulae as much as possible. To this end we will also use the primary constraints (4.21) and (4.22)—recall that at this moment we are still working with terms constituting the Hamiltonian  $H_0$  which is defined only for  $(\theta^A, p_B)$  satisfying the constraints.

**The term proportional to  $N^2$**  While gathering all the expressions proportional to  $N^2$  we see that some terms cancel at once and we get

$$\begin{aligned} N^2 \Big( & (*p_A \wedge \theta^B) \wedge (*p_B \wedge \theta^A) - \frac{1}{2}(*p_A \wedge \theta^A) \wedge (*p_B \wedge \theta^B) + \frac{1}{2}(p_A \wedge \theta^A) \wedge *(p_B \wedge \theta^B) + \\ & + 2p_A \wedge d\xi^A - (*p_A \wedge \theta^A) \wedge *(d\xi_B \wedge \theta^B) - (d\theta^A \wedge \xi_B) \wedge *(d\theta^B \wedge \xi_A) - \\ & - (d\xi^A \wedge \theta_A) \wedge *(d\theta^B \wedge \xi_B) + \frac{1}{2}(d\theta^A \wedge \xi_A) \wedge *(d\theta^B \wedge \xi_B) \Big) \quad (4.35) \end{aligned}$$

To simplify this expression note first that because  $\xi^A$  is a zero-form the sixth term in (4.35)

$$-(d\theta^A \wedge \xi_B) \wedge *(d\theta^B \wedge \xi_A) = -(d\theta^A \wedge \xi_A) \wedge *(d\theta^B \wedge \xi_B).$$

On the other hand by virtue of the constraint (4.22) the sixth and the last terms can be gather together with the second one giving

$$-(*p_A \wedge \theta^A) \wedge (*p_B \wedge \theta^B).$$

Moreover, due to the constraint (4.22) the fifth and the seventh terms cancel. Hence (4.35) can be written as

$$\begin{aligned} N^2 \Big( & (*p_A \wedge \theta^B) \wedge (*p_B \wedge \theta^A) - (*p_A \wedge \theta^A) \wedge (*p_B \wedge \theta^B) + \frac{1}{2}(p_A \wedge \theta^A) \wedge *(p_B \wedge \theta^B) + \\ & + 2p_A \wedge d\xi^A \Big). \quad (4.36) \end{aligned}$$

This expression can be further simplified—applying (4.27) to the first term of (4.36) and (4.28) to the second one we obtain

$$\begin{aligned} & (*p_A \wedge \theta^B) \wedge (*p_B \wedge \theta^A) - (*p_A \wedge \theta^A) \wedge (*p_B \wedge \theta^B) = -(\vec{\theta}^A \lrcorner *p_A)(\vec{\theta}^B \lrcorner *p_B)\epsilon + \\ & + (\vec{\theta}^A \lrcorner *p_B)(\vec{\theta}^B \lrcorner *p_A)\epsilon = -(\theta^A \wedge p_A) \wedge *(\theta^B \wedge p_B) + (\theta^B \wedge p_A) \wedge *(\theta^A \wedge p_B), \end{aligned}$$

where in the last step we used (4.9) and (4.24). Thus we arrived at a simple form of the term in (4.33) proportional to  $N^2$ :

$$N^2 \Big( (p_A \wedge \theta^B) \wedge *(p_B \wedge \theta^A) - \frac{1}{2}(p_A \wedge \theta^A) \wedge *(p_B \wedge \theta^B) + 2p_A \wedge d\xi^A \Big). \quad (4.37)$$

**The term proportional to  $N$**  Again while gathering all the expressions proportional to  $N$  some terms cancel at once and we get

$$N \left( 2(\vec{\theta}_A \lrcorner \mathcal{L}_{\vec{N}} \theta^A)(\vec{\theta}^B \lrcorner *p_B)\epsilon + 2(*p_A \wedge \theta^B) \wedge *(\mathcal{L}_{\vec{N}} \theta_B \wedge \theta^A) - 2(\vec{\theta}_A \lrcorner d\xi^A)(\vec{\theta}_B \lrcorner \mathcal{L}_{\vec{N}} \theta^B)\epsilon + \right. \\ \left. + 2d\xi_A \wedge *\mathcal{L}_{\vec{N}} \theta^A - (*p_A \wedge \theta^A) \wedge *(\mathcal{L}_{\vec{N}} \theta_B \wedge \theta^B) - 2(d\xi^A \wedge \theta_B) \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_A) + \right. \\ \left. + 2(d\theta^A \wedge \xi_B) \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_A) - (d\theta^A \wedge \xi_A) \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_B) \right) \quad (4.38)$$

Applying (4.27) to the second term of the expression above we see that the sum of the first and second terms can be expressed as

$$2(*p_A) \wedge *(\mathcal{L}_{\vec{N}} \theta^A) + 2(\xi^A *p_A) \wedge *(\xi_B \mathcal{L}_{\vec{N}} \theta^B) = 2\mathcal{L}_{\vec{N}} \theta^A \wedge p_A + 2(\xi^A *p_A) \wedge *(\xi_B \mathcal{L}_{\vec{N}} \theta^B)$$

On the other hand the seventh term in (4.38) can be transformed as follows

$$2(d\theta^A \wedge \xi_B) \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_A) = 2(\xi_B \mathcal{L}_{\vec{N}} \theta^B) \wedge \theta_A \wedge *d\theta^A = -2(\xi_B \mathcal{L}_{\vec{N}} \theta^B) we(\xi^A p_A) = \\ = -2(\xi_B \mathcal{L}_{\vec{N}} \theta^B) \wedge *(\xi^A *p_A) = -2(\xi^A *p_A) \wedge (\xi_B \mathcal{L}_{\vec{N}} \theta^B)$$

(here in the third step we used the constraint (4.21)). Thus the sum of the first, the second and the seventh term is simply

$$2\mathcal{L}_{\vec{N}} \theta^A \wedge p_A.$$

Moreover, the sum of the fifth and the last terms in (4.38) vanishes by virtue of the constraint (4.22):

$$-(*p_A \wedge \theta^A) \wedge *(\mathcal{L}_{\vec{N}} \theta_B \wedge \theta^B) - (d\theta^A \wedge \xi_A) \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_B) = - \\ -(\theta^A \wedge *p_A - \xi_A d\theta^A) \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_B) = 0.$$

Thus we managed to simplify (4.38) to

$$N \left( 2\mathcal{L}_{\vec{N}} \theta^A \wedge p_A - 2(\vec{\theta}_A \lrcorner d\xi^A)(\vec{\theta}_B \lrcorner \mathcal{L}_{\vec{N}} \theta^B)\epsilon + 2d\xi_A \wedge *\mathcal{L}_{\vec{N}} \theta^A - \right. \\ \left. - 2(d\xi^A \wedge \theta_B) \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_A) \right)$$

Now it is enough to apply (4.27) to the last term of the expression above to realize that (4.38) reduces to

$$2N\mathcal{L}_{\vec{N}} \theta^A \wedge p_A$$

which is the final expression of the terms in (4.33) proportional to  $N$ .

**The term independent of  $N$**  Gathering appropriate terms we obtain

$$\begin{aligned} & -2(\vec{\theta}_A \lrcorner \mathcal{L}_{\vec{N}} \theta^A)^2 \epsilon + \mathcal{L}_{\vec{N}} \theta_A \wedge \theta^B \wedge *(\mathcal{L}_{\vec{N}} \theta_B \wedge \theta^A) - \frac{1}{2} \mathcal{L}_{\vec{N}} \theta_A \wedge \theta^A \wedge *(\mathcal{L}_{\vec{N}} \theta_B \wedge \theta^B) + \\ & + 2(\vec{\theta}_C \lrcorner \mathcal{L}_{\vec{N}} \theta^C)^2 \epsilon - \mathcal{L}_{\vec{N}} \theta^A \wedge \theta_B \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_A) + \frac{1}{2} \mathcal{L}_{\vec{N}} \theta^A \wedge \theta_A \wedge *(\mathcal{L}_{\vec{N}} \theta^B \wedge \theta_B) = 0 \end{aligned}$$

In this way we arrived at the desired formula

$$\begin{aligned} \tilde{G}_{AB}(\rho^A, \rho^B) - E^A{}_B \wedge *E^B{}_A + \frac{1}{2} E^A{}_A \wedge *E^B{}_B &= -2N dN \wedge \theta_B \wedge *d\theta^B + \\ &+ N^2 \left( (p_A \wedge \theta^B) \wedge *(p_B \wedge \theta^A) - \frac{1}{2} (p_A \wedge \theta^A) \wedge *(p_B \wedge \theta^B) + 2p_A \wedge d\xi^A \right) + \\ &+ 2N \mathcal{L}_{\vec{N}} \theta^A \wedge p_A. \quad (4.39) \end{aligned}$$

Setting this to the Hamiltonian (4.32) we obtain

$$\begin{aligned} H[\theta^A, p_B, N, \vec{N}] &= \int_{\Sigma} -dN \wedge \theta_B \wedge *d\theta^B + N \left( \frac{1}{2} (p_A \wedge \theta^B) \wedge *(p_B \wedge \theta^A) - \right. \\ &- \frac{1}{4} (p_A \wedge \theta^A) \wedge *(p_B \wedge \theta^B) + p_A \wedge d\xi^A + \frac{1}{2} F^A{}_B \wedge *F^B{}_A - \frac{1}{4} F^A{}_A \wedge *F^B{}_B \Big) + \\ &+ \mathcal{L}_{\vec{N}} \theta^A \wedge p_A. \quad (4.40) \end{aligned}$$

What remains to be done is to remove the derivatives of the laps  $N$  and the shift  $\vec{N}$  appearing, respectively, in the first and in the last terms of the Hamiltonian above. Let us begin with the first term:

$$\begin{aligned} -dN \wedge \theta_B \wedge *d\theta^B &= -d(N\theta_B \wedge *d\theta^B) + Nd(\theta_B \wedge *d\theta^B) = \\ &= -d(N\theta_B \wedge *d\theta^B) - Nd(\xi^A p_A) = -d(N\theta_B \wedge *d\theta^B) - Nd\xi^A \wedge p_A - N\xi^A \wedge dp_A, \end{aligned}$$

where in the second step we applied the constraint (4.21). On the other hand it was shown in [17] that

$$(\mathcal{L}_{\vec{N}} \theta^A) \wedge p_A = -d\theta^A \wedge (\vec{N} \lrcorner p_A) - (\vec{N} \lrcorner \theta^A) \wedge dp_A + d((\vec{N} \lrcorner \theta^A) \wedge p_A). \quad (4.41)$$

Thus we arrive finally at the Hamiltonian  $H_0$  expressed explicitly as a function of the canonical variables, the laps and the shift

$$\begin{aligned} H_0[\theta^A, p_B, N, \vec{N}] &= \int_{\Sigma} N \left( \frac{1}{2} (p_A \wedge \theta^B) \wedge *(p_B \wedge \theta^A) - \frac{1}{4} (p_A \wedge \theta^A) \wedge *(p_B \wedge \theta^B) - \right. \\ &- \xi^A \wedge dp_A + \frac{1}{2} (d\theta_A \wedge \theta^B) \wedge *(d\theta_B \wedge \theta^A) - \frac{1}{4} (d\theta_A \wedge \theta^A) \wedge *(d\theta_B \wedge \theta^B) \Big) - \\ &- d\theta^A \wedge (\vec{N} \lrcorner p_A) - (\vec{N} \lrcorner \theta^A) \wedge dp_A, \quad (4.42) \end{aligned}$$

which is exactly the Hamiltonian (3.9). In order to extend  $H_0$  to the whole phase space we add to it the smeared primary constraints (3.7) and (3.8) and arrive thereby at (3.10).

The Hamiltonian (3.10) depends on the Lagrange multipliers  $N$  and  $\vec{N}$ . Variations of the Hamiltonian with respect to the multipliers give us the secondary constraints (3.11). Expressing the r.h.s. of this equation by means of the smeared versions (3.12) and (3.13) of, respectively, the scalar and the vector constraints we obtain (3.14) and (3.15).

## 5 Derivation of the constraint algebra

In this subsection we will calculate Poisson brackets of all the constraints proving thereby the results presented in Subsection 3.2.

If  $F$  and  $G$  are functionals on the phase space then their Poisson bracket [5]

$$\{F, G\} = \int_{\Sigma} \left( \frac{\delta F}{\delta \theta^A} \wedge \frac{\delta G}{\delta p_A} - \frac{\delta G}{\delta \theta^A} \wedge \frac{\delta F}{\delta p_A} \right)$$

(for a definition of a functional derivative with respect to a differential form see [17]).

Calculating functional derivatives with respect to  $\theta^A$  and  $p_A$  of the smeared constraints would be straightforward if the Hodge operator  $*$  did not depend on  $\theta^A$ . This dependence gives raise to a rather complicated expression—if  $k$ -forms  $\alpha$  and  $\beta$  do not depend on the canonical variables then [17]

$$\frac{\delta}{\delta \theta^A} \int_{\Sigma} \alpha \wedge * \beta = \vec{\theta}^B \lrcorner [\eta_{AB} \alpha \wedge * \beta - (\vec{\theta}_A \lrcorner \alpha) \wedge * (\vec{\theta}_B \lrcorner \beta) - (\vec{\theta}_B \lrcorner \alpha) \wedge * (\vec{\theta}_A \lrcorner \beta)]. \quad (5.1)$$

It will be convenient to introduce a short notation for the r.h.s. of this equation:

$$\vec{\theta}^B \lrcorner [\eta_{AB} \alpha \wedge * \beta - (\vec{\theta}_A \lrcorner \alpha) \wedge * (\vec{\theta}_B \lrcorner \beta) - (\vec{\theta}_B \lrcorner \alpha) \wedge * (\vec{\theta}_A \lrcorner \beta)] \equiv \alpha \wedge *'_A \beta. \quad (5.2)$$

Let us emphasize that the symbol  $\alpha \wedge *'_A \beta$  will also be used as an abbreviation of the l.h.s. of (5.2) in those cases when the forms  $\alpha$  and  $\beta$  do depend on the canonical variables. An important property of the two-form  $\alpha \wedge *'_A \beta$  is that it vanishes once contracted with the function  $\xi^A$  [17]:

$$\xi^A (\alpha \wedge *'_A \beta) = 0 \quad (5.3)$$

(this is true for any  $\alpha$  and  $\beta$  including cases when the forms depend on the canonical variables).

### 5.1 Tensor calculus

Although our original wish was to carry out all necessary calculations using differential form calculus only we were forced in some cases to use tensor calculus. Below we gathered some expressions which will be applied repeatedly in the sequel.

Let  $\nabla_a$  denote a covariant derivative on  $\Sigma$  defined by the Levi-Civita connection given by the metric  $q$ . Consequently, by virtue of (2.9)

$$0 = \nabla_i q_{jk} = \nabla_i (\theta_{Aj} \theta_k^A) = (\nabla_i \theta_{Aj}) \theta_k^A + \theta_{Aj} (\nabla_i \theta_k^A) \quad (5.4)$$

and

$$(\nabla_a \theta^{Bb}) \theta_{Bb} = 0. \quad (5.5)$$

Note also that

$$\nabla_a \epsilon_{ijk} = 0 \quad (5.6)$$

because  $\epsilon$  is defined by  $q$ .

For any one-forms  $\alpha$  and  $\beta$

$$\begin{aligned} d\alpha &= \nabla_a \alpha_b dx^a \wedge dx^b, & *d\alpha &= (\nabla_a \alpha_b) \epsilon^{ab}{}_c dx^c, \\ d\alpha \wedge \beta &= (\nabla_a \alpha_b) \beta_c dx^a \wedge dx^b \wedge dx^c, & *(d\alpha \wedge \beta) &= (\nabla_a \alpha_b) \beta_c \epsilon^{abc}, \end{aligned} \quad (5.7)$$

If  $\alpha$  is a one-form,  $\beta$  a two-form and  $\gamma$  a three-form then

$$\begin{aligned} *d*\alpha &= q^{ab} \nabla_a \alpha_b = \nabla^a \alpha_a, \\ *d*\beta &= q^{ab} \nabla_a \beta_{cb} dx^c = \nabla^b \beta_{cb} dx^c, \\ *d*\gamma &= q^{ab} \nabla_a \gamma_{dcb} dx^d \otimes dx^c = \nabla^b \gamma_{dcb} dx^d \otimes dx^c = \frac{1}{2} \nabla^b \gamma_{dcb} dx^d \wedge dx^c, \end{aligned} \quad (5.8)$$

We will also apply the following identities (for a proof see e.g. [17]):

$$\epsilon^{abc} \epsilon_{ibc} = 2\delta^a{}_i, \quad \epsilon^{abc} \epsilon_{ijc} = 2\delta^{[a}{}_i \delta^{b]}{}_j, \quad \epsilon_{ijk} \epsilon^{abc} = 3! \delta^{[a}{}_i \delta^b{}_j \delta^{c]}{}_k. \quad (5.9)$$

and a formula defining the Hodge operator:

$$\alpha \wedge *\beta = \frac{1}{k!} \alpha_{a_1 \dots a_k} \beta^{a_1 \dots a_k} \epsilon, \quad (5.10)$$

where  $\alpha$  and  $\beta$  are  $k$ -forms.

## 5.2 Poisson brackets of $B(a)$ and $R(b)$

In this subsection we will calculate Poisson brackets  $\{B(a), B(a')\}$ ,  $\{R(b), R(b')\}$  and  $\{B(a), R(b)\}$ .

### 5.2.1 Auxiliary formulae

The following formulas will be used while calculating the brackets:

$$b \wedge \alpha \wedge *(b' \wedge \beta) - (b \leftrightarrow b') = *(b \wedge b') \wedge \alpha \wedge \beta, \quad (5.11)$$

$$(\alpha \wedge *_A' \beta) \wedge *(b \wedge \theta^A) = 0, \quad (5.12)$$

$$\theta_A \wedge *(\alpha \wedge \theta^A) = (3 - k) * \alpha, \quad (5.13)$$

$$\epsilon_{D B C A} \theta^B \wedge \theta^C \wedge *(b \wedge \theta^A) = 0, \quad (5.14)$$

$$-\frac{1}{2} \epsilon^D{}_{B C A} \theta^B \wedge \theta^C \xi^A = * \theta^D. \quad (5.15)$$

In the first formula  $\alpha, \beta, b, b'$  are one-forms, in the second one  $\alpha$  and  $\beta$  are  $k$ -forms, and  $b$  is a one-form, in the third one  $\alpha$  is a  $k$ -form, and in the fourth one  $b$  is a one-form.

*Proof of (5.11).*

$$\begin{aligned} b \wedge \alpha \wedge *(b' \wedge \beta) - (b \leftrightarrow b') &= b \wedge **[\alpha \wedge *(b' \wedge \beta)] - (b \leftrightarrow b') = \\ &= -b \wedge *[(\vec{\alpha} \lrcorner b')\beta - b' \vec{\alpha} \lrcorner \beta] - (b \leftrightarrow b') = \vec{\alpha} \lrcorner (b \wedge b') \wedge * \beta = -(b \wedge b') \vec{\alpha} \lrcorner * \beta = \\ &= (b \wedge b') \wedge *(\alpha \wedge \beta) = *(b \wedge b') \wedge (\alpha \wedge \beta), \end{aligned} \quad (5.16)$$

where in the second step we used (4.9), in the third the fact that  $b \wedge *b'$  is symmetric in  $b$  and  $b'$  and finally in the fifth step we used (4.9) again.  $\square$

*Proof of (5.12).* To prove (5.12) note that the two-form  $\alpha \wedge *_A' \beta$  given by (5.2) is of the form  $\vec{\theta}^B \lrcorner \gamma_{AB}$ , where the three-form  $\gamma_{AB}$  is symmetric:  $\gamma_{AB} = \gamma_{BA}$ . Thus

$$(\alpha \wedge *_A' \beta) \wedge *(b \wedge \theta^A) = (\vec{\theta}^B \lrcorner \gamma_{AB}) \wedge *(b \wedge \theta^A) = \gamma_{AB} \wedge \vec{\theta}^B \lrcorner *(b \wedge \theta^A) = \gamma_{AB} \wedge *(b \wedge \theta^A \wedge \theta^B),$$

where in the last step we used (4.9). But  $(\theta^B \wedge b \wedge \theta^A)$  is antisymmetric in  $AB$ , hence (5.12) follows.  $\square$

*Proof of (5.13).*

$$\theta_A \wedge *(\alpha \wedge \theta^A) = \theta_A \wedge \vec{\theta}^A \lrcorner * \alpha = (3 - k) * \alpha,$$

where in the first step we used (4.9) and in the second one we applied (4.17).  $\square$

*Proof of (5.14).* Let us transform the following term by means of (4.27):

$$\theta^B \wedge \theta^C \wedge *(b \wedge \theta^A) = -(\vec{\theta}^A \lrcorner \theta^B)(\vec{\theta}^C \lrcorner b) \epsilon + (\vec{\theta}^A \lrcorner \theta^C) \theta^B \wedge b. \quad (5.17)$$

Note that the first term at the r.h.s. of the equation above is symmetric in  $AB$ , while the second one—in  $AC$ . This means that both terms vanish once contracted with  $\epsilon_{D B C A}$ .  $\square$

The last formula (5.15) is proven in [17].

### 5.2.2 Poisson bracket $\{B(a), B(a')\}$

Let

$$B_1(a) := \int_{\Sigma} a \wedge \theta^A \wedge *d\theta_A, \quad B_2(a) := \int_{\Sigma} a \wedge \xi^A p_A = \int_{\Sigma} (*\xi^A) \wedge *(a \wedge p_A).$$

Taking into account (3.7) we see that  $B(a) = B_1(a) + B_2(a)$  and consequently

$$\{B(a), B(a')\} = \{B_1(a), B_1(a')\} + \left( \{B_1(a), B_2(a')\} - \{B_1(a'), B_2(a)\} \right) + \{B_2(a), B_2(a')\}. \quad (5.18)$$

The corresponding variational derivatives read

$$\begin{aligned} \frac{\delta B_1(a)}{\delta \theta^A} &= -a \wedge *d\theta_A + (a \wedge \theta^B) \wedge *_A' d\theta_B + d*(a \wedge \theta_A), \\ \frac{\delta B_1(a)}{\delta p_A} &= 0, \\ \frac{\delta B_2(a)}{\delta \theta^A} &= -\frac{1}{2} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge *(a \wedge p_D) + (*\xi^B) \wedge *_A' (a \wedge p_B), \\ \frac{\delta B_2(a)}{\delta p_A} &= a \xi^A. \end{aligned} \quad (5.19)$$

Obviously,  $\{B_1(a), B_1(a')\} = 0$ . The next term in (5.18)

$$\begin{aligned} \{B_1(a), B_2(a')\} - \{B_1(a'), B_2(a)\} &= \int_{\Sigma} -a \wedge *d\theta_A \wedge a' \xi^A + d*(a \wedge \theta_A) \wedge a' \xi^A - (a \leftrightarrow a') = \\ &= \int_{\Sigma} 2*(a \wedge a') \xi^A d\theta_A - \left( a \wedge \theta_A \wedge *(a' \wedge d\xi^A) - (a \leftrightarrow a') \right) = \int_{\Sigma} 2*(a \wedge a') \xi^A d\theta_A - \\ &\quad - *(a \wedge a') \theta_A \wedge d\xi^A = \int_{\Sigma} *(a \wedge a') \xi^A d\theta_A, \end{aligned}$$

where in the first step we used (5.3) and in the second one (5.11). The last term in (5.18)

$$\{B_2(a), B_2(a')\} = \int_{\Sigma} -\frac{1}{2} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge *(a \wedge p_D) \wedge a' \xi^A - (a \leftrightarrow a').$$

By virtue of (5.15) and (4.9)

$$-\frac{1}{2} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge *(a \wedge p_D) \wedge a' \xi^A = *\theta_D \wedge a' *(a \wedge p^D) = a \wedge p_D \vec{\theta}^D \lrcorner a'.$$

Thus

$$\begin{aligned} \{B_2(a), B_2(a')\} &= - \int_{\Sigma} \vec{\theta}^D \lrcorner (a \wedge a') p_D = \int_{\Sigma} a \wedge a' \wedge \vec{\theta}^D \lrcorner p_D = \int_{\Sigma} -a \wedge a' \wedge *(\theta^D \wedge *p_D) = \\ &= \int_{\Sigma} -(a \wedge a') \theta^A \wedge *p_A, \end{aligned}$$



where in the third step we used (4.9).

We conclude that

$$\{B(a), B(a')\} = - \int_{\Sigma} *(a \wedge a') \wedge (\theta^A \wedge *p_A - \xi^A d\theta_A) = -R(*(a \wedge a')).$$

### 5.2.3 Poisson bracket $\{R(b), R(b')\}$

Taking into account (3.8) we define

$$R_1(b) := \int_{\Sigma} b \wedge \theta^A \wedge *p_A, \quad R_2(b) := \int_{\Sigma} b \wedge \xi^A d\theta_A.$$

Then

$$\begin{aligned} \{R(b), R(b')\} &= \{R_1(b), R_1(b')\} - \left( \{R_1(b), R_2(b')\} - \{R_1(b'), R_2(b)\} \right) + \\ &\quad + \{R_2(b), R_2(b')\}. \end{aligned} \quad (5.20)$$

The corresponding variational derivatives read

$$\begin{aligned} \frac{\delta R_1(b)}{\delta \theta^A} &= -b \wedge *p_A + (b \wedge \theta^B) \wedge *_A p_B \\ \frac{\delta R_1(b)}{\delta p_A} &= *(b \wedge \theta^A), \\ \frac{\delta R_2(b)}{\delta \theta^A} &= d(b\xi_A) + (b \wedge d\theta^B) \wedge *_A (*\xi_B) - \frac{1}{2}\epsilon_{D B C A} \theta^B \wedge \theta^C *(b \wedge d\theta^D) \\ \frac{\delta R_2(b)}{\delta p_A} &= 0. \end{aligned} \quad (5.21)$$

Due to (5.12) the first bracket at the r.h.s. of (5.20) reduces to

$$\{R_1(b), R_1(b')\} = \int_{\Sigma} -b \wedge *p_A \wedge *(b' \wedge \theta^A) - (b \leftrightarrow b') = \int_{\Sigma} *(b \wedge b') \wedge \theta^A \wedge *p_A$$

(here in the last step we used (5.11)). Similarly, by virtue of (5.12) the next two brackets in (5.20) reduce to

$$\begin{aligned} \{R_1(b), R_2(b')\} - \{R_1(b'), R_2(b)\} &= \\ &= \int_{\Sigma} [-d(b'\xi_A) + \frac{1}{2}\epsilon_{D B C A} \theta^B \wedge \theta^C *(b' \wedge d\theta^D)] \wedge *(b \wedge \theta^A) - (b \leftrightarrow b') = \\ &= \int_{\Sigma} b' \wedge d\xi^A \wedge *(b \wedge \theta^A) + \frac{1}{2}\epsilon_{D B C A} \theta^B \wedge \theta^C \wedge *(b \wedge \theta^A) *(b' \wedge d\theta^D) - (b \leftrightarrow b'). \end{aligned}$$

Note now that due to (5.14) the term containing  $\epsilon_{DBCA}$  vanishes. Therefore

$$\{R_1(b), R_2(b')\} - \{R_1(b'), R_2(b)\} = \int_{\Sigma} *(b \wedge b') \wedge \xi^A d\theta_A,$$

where we applied (5.11). Because  $\{R_2(b), R_2(b')\} = 0$  we finally obtain

$$\{R(b), R(b')\} = \int_{\Sigma} *(b \wedge b') \wedge (\theta^A \wedge *p_A - \xi^A d\theta_A) = R(*(b \wedge b')).$$

#### 5.2.4 Poisson bracket $\{B(a), R(b)\}$

Clearly,

$$\begin{aligned} \{B(a), R(b)\} &= \{B_1(a), R_1(b)\} + \{B_2(a), R_1(b)\} - \{B_1(a), R_2(b)\} - \\ &\quad - \{B_2(a), R_2(b)\}. \end{aligned} \quad (5.22)$$

Using (5.12) we immediately obtain

$$\begin{aligned} \{B_1(a), R_1(b)\} &= \int_{\Sigma} -a \wedge *d\theta_A \wedge *(b \wedge \theta^A) + d*(a \wedge \theta_A) \wedge *(b \wedge \theta^A) = \\ &= \int_{\Sigma} *(a \wedge b) \wedge \theta^A \wedge *d\theta_A - b \wedge *d\theta_A \wedge *(a \wedge \theta^A) + d*(a \wedge \theta_A) \wedge *(b \wedge \theta^A), \end{aligned}$$

where we used (5.11) to transform the first term. The next bracket in (5.22)

$$\begin{aligned} \{B_2(a), R_1(b)\} &= \int_{\Sigma} -\frac{1}{2}\epsilon^D_{BCA}\theta^B \wedge \theta^C *(a \wedge p_D) \wedge *(b \wedge \theta^A) + b \wedge *p_A \wedge a\xi^A = \\ &= \int_{\Sigma} *(a \wedge b) \wedge \xi^A p_A, \end{aligned}$$

where the term containing  $\epsilon^D_{BCA}$  vanishes due to (5.17). The bracket  $\{B_1(a), R_2(b)\}$  is obviously zero and

$$\begin{aligned} -\{B_2(a), R_2(b)\} &= \int_{\Sigma} [(d(b\xi_A)) - \frac{1}{2}\epsilon_{DBCA}\theta^B \wedge \theta^C *(b \wedge d\theta^D)] \wedge a\xi^A = \\ &= \int_{\Sigma} -db \wedge a + *\theta_D \wedge a \wedge *(b \wedge d\theta^D) = \int_{\Sigma} -db \wedge a + *a \wedge \theta_D \wedge *(db \wedge \theta^D) - \theta_D \wedge *a \wedge *(b \wedge \theta^D) = \\ &= \int_{\Sigma} *d*(\theta_D \wedge *a) \wedge *(b \wedge \theta^D), \end{aligned}$$

where in the first step we used (5.3), in the second one we applied (5.15) and in the last step (5.13). Gathering the partial results we obtain

$$\begin{aligned} \{B(a), R(b)\} &= \int_{\Sigma} *(a \wedge b)(\theta^A \wedge *d\theta_A + \xi^A p_A) - b \wedge *d\theta_A \wedge *(a \wedge \theta^A) + \\ &\quad + [d*(a \wedge \theta_A) + *d*(\theta_A \wedge *a)] \wedge *(b \wedge \theta^A) = B(*(a \wedge b)) + \\ &\quad + \int_{\Sigma} -b \wedge *d\theta_A \wedge *(a \wedge \theta^A) + [d*(a \wedge \theta_A) + *d*(\theta_A \wedge *a)] \wedge *(b \wedge \theta^A). \end{aligned} \quad (5.23)$$

Let us show now that the last two terms in (5.23) give zero. The first of them can be expressed as follows:

$$\begin{aligned} -b \wedge *d\theta_A \wedge *(a \wedge \theta^A) &= -[b \wedge *d*(\theta_A)] \wedge *[a \wedge \theta^A] = -\frac{1}{2}[b \wedge *d*(\theta_A)]_{lj}[a \wedge \theta^A]^{lj}\epsilon \\ &= b_l(\nabla^k \theta_A^i) \epsilon_{ijk}(a^j \theta^{Al} - a^l \theta^{Aj})\epsilon = \delta_n^l b^n (\nabla^k \theta_A^i) \epsilon_{ijk}(a^j \theta_l^A - a_l \theta^{Aj})\epsilon \end{aligned}$$

(the second equality holds by virtue of (5.10) and the third one due to (5.8) and (5.6)). Using (5.9) we rewrite  $\delta_n^l$  by means of “epsilons” and continue transformations

$$\begin{aligned} -b \wedge *d\theta_A \wedge *(a \wedge \theta^A) &= \frac{1}{2} \epsilon^{lab} \epsilon_{nab} b^n (\nabla^k \theta_A^i) \epsilon_{ijk}(a^j \theta_l^A - a_l \theta^{Aj})\epsilon = \\ &= \epsilon_{nab} (\delta_i^l \delta_j^a \delta_k^b + \delta_i^a \delta_j^b \delta_k^l + \delta_i^b \delta_j^l \delta_k^a) b^n (\nabla^k \theta_A^i) (a^j \theta_l^A - a_l \theta^{Aj})\epsilon = \\ &= -a_i b^n \epsilon_{nj k} (\nabla^k \theta_A^i) \theta^{Aj} \epsilon + a^j b^n \epsilon_{nij} (\nabla^k \theta_A^i) \theta_k^A \epsilon - a_k b^n \epsilon_{nij} (\nabla^k \theta_A^i) \theta^{Aj} \epsilon = \\ &= a^j b^n \epsilon_{jni} (\nabla^k \theta_A^i) \theta_k^A \epsilon + a_k b^n \epsilon_{nij} \theta^{Aj} (\nabla^i \theta_A^k - \nabla^k \theta_A^i) \epsilon \end{aligned}$$

—here in the second step we applied (5.9) to express  $\epsilon^{lab} \epsilon_{ijk}$  by means of “deltas”.

On the other hand the last term in (5.23)

$$\begin{aligned} [**d*(a \wedge \theta_A) + *d*(\theta_A \wedge *a)] \wedge *(b \wedge \theta^A) &= \\ &= [\nabla_j(a^i \theta^{Aj} - a^j \theta^{Ai}) + \nabla^i(\theta^{Aj} a_j)] \epsilon_{ikl} b^k \theta_A^l \epsilon = \\ &= a^i b^k \epsilon_{ikl} \theta_A^l (\nabla^j \theta_j^A) \epsilon + a_j b^k \epsilon_{ikl} \theta^{Al} (\nabla^i \theta_A^j - \nabla^j \theta_A^i) \epsilon \end{aligned}$$

—the second equality holds by virtue of (5.10) and (5.6).

Thus the sum of the last two terms in (5.23) is equal to

$$a^i b^k \epsilon_{ikl} [\theta_j^A (\nabla^j \theta_A^l) + \theta_A^l (\nabla^j \theta_j^A)] \epsilon = 0,$$

where we used (5.4). Finally

$$\{B(a), R(b)\} = B(*(a \wedge b)).$$

### 5.3 Poisson bracket of $S(M)$ and $S(M')$

Following [17] we split  $S(M)$  (defined by (3.12)) into three functionals

$$\begin{aligned} S_1(M) &:= \int_{\Sigma} M \left[ \frac{1}{2} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) - \frac{1}{4} (p_A \wedge \theta^A) \wedge * (p_B \wedge \theta^B) \right] \\ S_2(M) &:= - \int_{\Sigma} M \xi^A dp_A, \\ S_3(M) &:= \int_{\Sigma} M \left[ \frac{1}{2} (d\theta_A \wedge \theta^B) \wedge * (d\theta_B \wedge \theta^A) - \frac{1}{4} (d\theta_A \wedge \theta^A) \wedge * (d\theta_B \wedge \theta^B) \right]. \end{aligned} \quad (5.24)$$

Then

$$\begin{aligned} \{S(M), S(M')\} &= \{S_1(M), S_1(M')\} + \{S_2(M), S_2(M')\} + \{S_3(M), S_3(M')\} + \\ &\quad \left( \{S_1(M), S_2(M')\} + \{S_2(M), S_3(M')\} + \{S_3(M), S_1(M')\} - (M \leftrightarrow M') \right). \end{aligned} \quad (5.25)$$

#### 5.3.1 Poisson brackets $\{S_i(M), S_j(M')\}$

Functional derivatives of the functionals read:

$$\begin{aligned} \frac{\delta S_1(M)}{\delta \theta^A} &= M \left( p_B * (p_A \wedge \theta^B) - \frac{1}{2} p_A * (p_B \wedge \theta^B) + \right. \\ &\quad \left. + \frac{1}{2} (p_C \wedge \theta^B) \wedge *_A' (p_B \wedge \theta^C) - \frac{1}{4} (p_B \wedge \theta^B) \wedge *_A' (p_C \wedge \theta^C) \right), \\ \frac{\delta S_1(M)}{\delta p_A} &= M \left( \theta^B * (p_B \wedge \theta^A) - \frac{1}{2} \theta^A * (p_B \wedge \theta^B) \right), \\ \frac{\delta S_2(M)}{\delta \theta^A} &= M \left( \frac{1}{2} (*dp_D) \epsilon^D_{BCA} \theta^B \wedge \theta^C - (*\xi^B) \wedge *_A' dp_B \right), \\ \frac{\delta S_2(M)}{\delta p_A} &= d(M \xi^A), \\ \frac{\delta S_3(M)}{\delta \theta^A} &= d \left( M \theta^B * (d\theta_B \wedge \theta_A) - \frac{M}{2} \theta_A * (d\theta_B \wedge \theta^B) \right) + M \left( d\theta_B * (d\theta_A \wedge \theta^B) - \right. \\ &\quad \left. - \frac{1}{2} d\theta_A * (d\theta_B \wedge \theta^B) + \frac{1}{2} (d\theta_C \wedge \theta^B) \wedge *_A' (d\theta_B \wedge \theta^C) - \right. \\ &\quad \left. - \frac{1}{4} (d\theta_B \wedge \theta^B) \wedge *_A' (d\theta_C \wedge \theta^C) \right), \\ \frac{\delta S_3(M)}{\delta p_A} &= 0. \end{aligned}$$

Let us begin the calculations with the bracket  $\{S_1(M), S_1(M')\}$ :

$$\{S_1(M), S_1(M')\} = \int_{\Sigma} \frac{\delta S_1(M)}{\delta \theta^A} \wedge \frac{\delta S_1(M')}{\delta p_A} - (M \leftrightarrow M') = 0$$

because the first term under the integral is symmetric in  $M$  and  $M'$ . It was shown in [17] that

$$\{S_2(M), S_2(M')\} = - \int_{\Sigma} (\vec{m} \lrcorner \theta^A) \wedge dp_A, \quad (5.26)$$

where

$$m := M dM' - M' dM.$$

Because  $S_3(M)$  does not depend on the momentum

$$\{S_3(M), S_3(M')\} = 0.$$

Next,

$$\begin{aligned} \{S_1(M), S_2(M')\} - (M \leftrightarrow M') &= \int_{\Sigma} M [p_B * (p_A \wedge \theta^B) - \frac{1}{2} p_A * (p_B \wedge \theta^B)] \wedge \xi^A dM' - \\ &- (M \leftrightarrow M') = \int_{\Sigma} m \wedge p_B * (\xi^A p_A \wedge \theta^B) - \frac{1}{2} m \wedge \xi^A p_A * (p_B \wedge \theta^B), \end{aligned} \quad (5.27)$$

where in the second step we removed terms symmetric in  $M$  and  $M'$ .

$$\begin{aligned} \{S_2(M), S_3(M')\} - (M \leftrightarrow M') &= \int_{\Sigma} -M' [d\theta_B * (d\theta_A \wedge \theta^B) - \frac{1}{2} d\theta_A * (d\theta_B \wedge \theta^B)] \wedge \xi^A dM - \\ &- (M \leftrightarrow M') = \int_{\Sigma} m \wedge d\theta_B * (\xi^A d\theta_A \wedge \theta^B) - \frac{1}{2} m \wedge \xi^A d\theta_A * (d\theta_B \wedge \theta^B), \end{aligned} \quad (5.28)$$

where in the second step there vanished terms symmetric in  $M$  and  $M'$  and a total derivative.

$$\begin{aligned} \{S_3(M), S_1(M')\} - (M \leftrightarrow M') &= \int_{\Sigma} dM [\theta^B * (d\theta_B \wedge \theta_A) - \frac{1}{2} \theta_A * (d\theta_B \wedge \theta^B)] \wedge \\ &\wedge M' [\theta^B * (p_B \wedge \theta^A) - \frac{1}{2} \theta^A * (p_B \wedge \theta^B)] - (M \leftrightarrow M') = \\ &= \int_{\Sigma} -m \wedge [\theta^B \wedge \theta^C * (d\theta_B \wedge \theta_A * (p_C \wedge \theta^A)) - \frac{1}{2} \theta_A * (p_C \wedge \theta^A) \wedge \theta^C * (d\theta_B \wedge \theta^B) - \\ &- \frac{1}{2} \theta^B \wedge \theta^A * (d\theta_B \wedge \theta_A) * (p_C \wedge \theta^C)] = \int_{\Sigma} -m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge * p_A) - \\ &- \frac{1}{2} m \wedge \theta^A \wedge * p_A * (d\theta_B \wedge \theta^B) + \frac{1}{2} m \wedge \theta^B \wedge * d\theta_B * (p_A \wedge \theta^A), \end{aligned} \quad (5.29)$$

where in the second step some terms vanish by virtue of their symmetricity in  $M$  and  $M'$  and the last step we used (5.13).

Thus we obtain explicite expressions for all the terms at the r.h.s. of (5.25):

$$\begin{aligned} \{S(M), S(M')\} = & \int_{\Sigma} -(\vec{m} \lrcorner \theta^A) \wedge dp_A + m \wedge p_B * (\xi^A p_A \wedge \theta^B) - \frac{1}{2} m \wedge \xi^A p_A * (p_B \wedge \theta^B) + \\ & + m \wedge d\theta_B * (\xi^A d\theta_A \wedge \theta^B) - \frac{1}{2} m \wedge \xi^A d\theta_A * (d\theta_B \wedge \theta^B) - m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge *p_A) - \\ & - \frac{1}{2} m \wedge \theta^A \wedge *p_A * (d\theta_B \wedge \theta^B) + \frac{1}{2} m \wedge \theta^B \wedge *d\theta_B * (p_A \wedge \theta^A). \end{aligned} \quad (5.30)$$

### 5.3.2 Isolating constraints

Now our goal is to show that the r.h.s. of (5.30) is a sum of the smeared primary and secondary constraints with appropriately chosen smearing fields.

It is easy to see that

$$\{S_2(M), S_2(M')\} = V(\vec{m}) + \int_{\Sigma} d\theta^A \wedge \vec{m} \lrcorner p_A.$$

Similarly,

$$\begin{aligned} \{S_1(M), S_2(M')\} - (M \leftrightarrow M') = & \int_{\Sigma} [\theta^B * (m \wedge p_B) - \frac{1}{2} m * (p_B \wedge \theta^B)] \wedge \xi^A p_A = \\ = & B\left(\theta^B * (m \wedge p_B) - \frac{1}{2} m * (p_B \wedge \theta^B)\right) + \int_{\Sigma} - * (m \wedge p_B) \theta^B \wedge \theta^A \wedge *d\theta_A + \\ & + \frac{1}{2} * (p_B \wedge \theta^B) m \wedge \theta^A \wedge *d\theta_A \end{aligned}$$

and

$$\begin{aligned} \{S_2(M), S_3(M')\} - (M \leftrightarrow M') = & \int_{\Sigma} [* (m \wedge d\theta_B) \theta^B - \frac{1}{2} m * (d\theta_B \wedge \theta^B)] \wedge \xi^A d\theta_A = \\ = & -R\left(\theta^B * (m \wedge d\theta_B) - \frac{1}{2} m * (d\theta_B \wedge \theta^B)\right) + \int_{\Sigma} * (m \wedge d\theta_B) \theta^B \wedge \theta^A \wedge *p_A - \\ & - \frac{1}{2} * (d\theta_B \wedge \theta^B) m \wedge \theta^A \wedge *p_A. \end{aligned}$$

Thus (5.30) can be rewritten as follows:

$$\begin{aligned} \{S(M), S(M')\} = & V(\vec{m}) + B\left(\theta^B * (m \wedge p_B) - \frac{1}{2} m * (p_B \wedge \theta^B)\right) - \\ & - R\left(\theta^B * (m \wedge d\theta_B) - \frac{1}{2} m * (d\theta_B \wedge \theta^B)\right) + \text{remaining terms}, \end{aligned} \quad (5.31)$$

where the remaining terms read

$$\begin{aligned} \int_{\Sigma} & - * (m \wedge p_B) \theta^B \wedge \theta^A \wedge * d\theta_A + * (m \wedge d\theta_B) \theta^B \wedge \theta^A \wedge * p_A + \\ & + m \wedge \theta^A \wedge * d\theta_A * (p_B \wedge \theta^B) - m \wedge \theta^A \wedge * p_A * (d\theta_B \wedge \theta^B) - \\ & - m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge * p_A) + d\theta^A \wedge \vec{m} \lrcorner p_A. \end{aligned} \quad (5.32)$$

Now let us get rid of the last term in (5.32) which by virtue of (4.9) can be expressed as follows [17]:

$$d\theta^A \wedge \vec{m} \lrcorner p_A = d\theta^A \wedge * (p_A \wedge m) = * p_A \wedge m \wedge * d\theta^A = m \wedge * d\theta^A \wedge * p_A.$$

To this end we transform a half of the fourth term in (5.32):

$$\begin{aligned} -\frac{1}{2} m \wedge \theta^A \wedge * p_A * (d\theta_B \wedge \theta^B) &= -\frac{1}{2} m \wedge \theta^A \wedge * p_A \vec{\theta}^B \lrcorner * d\theta_B = -\frac{1}{2} \vec{\theta}^B \lrcorner (m \wedge \theta^A \wedge * p_A) \wedge * d\theta_B = \\ &= -\frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * p_A \wedge * d\theta_B + \frac{1}{2} m \wedge (\eta^{BA} + \xi^B \xi^A) * p_A \wedge * d\theta_B - \\ &\quad - \frac{1}{2} m \wedge \theta^A (\vec{\theta}^B \lrcorner * p_A) \wedge * d\theta_B = -\frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * p_A \wedge * d\theta_B + \\ &\quad + \frac{1}{2} m \wedge * p_A \wedge * d\theta^A + \frac{1}{2} m \wedge * (\xi^A p_A) \wedge * (\xi^B d\theta_B) - \frac{1}{2} m \wedge \theta^A \wedge * d\theta_B * (p_A \wedge \theta^B), \end{aligned}$$

where in the first step we used (4.9), in the third one (4.23) and in the last one (4.9) again. Similarly, a half of the third term in (5.32)

$$\begin{aligned} \frac{1}{2} m \wedge \theta^A \wedge * d\theta_A * (p_B \wedge \theta^B) &= \frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * d\theta_A \wedge * p_B + \\ &\quad - \frac{1}{2} m \wedge * d\theta_A \wedge * p^A - \frac{1}{2} m \wedge * (\xi^A d\theta_A) \wedge * (\xi^B p_B) + \frac{1}{2} m \wedge \theta^A \wedge * p_B * (d\theta_A \wedge \theta^B). \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{2} m \wedge \theta^A \wedge * d\theta_A * (p_B \wedge \theta^B) &- \frac{1}{2} m \wedge \theta^A \wedge * p_A * (d\theta_B \wedge \theta^B) + d\theta^A \wedge \vec{m} \lrcorner p_A = \\ &= -\frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * p_A \wedge * d\theta_B + \frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * d\theta_A \wedge * p_B - \\ &- \frac{1}{2} m \wedge \theta^A \wedge * d\theta_B * (p_A \wedge \theta^B) + \frac{1}{2} m \wedge \theta^A \wedge * p_B * (d\theta_A \wedge \theta^B) + m \wedge * (\xi^A p_A) \wedge * (\xi^B d\theta_B). \end{aligned}$$

and the terms (5.32) can be now expressed as

$$\begin{aligned}
& \int_{\Sigma} - * (m \wedge p_B) \theta^B \wedge \theta^A \wedge * d\theta_A + * (m \wedge d\theta_B) \theta^B \wedge \theta^A \wedge * p_A + \\
& + \frac{1}{2} m \wedge \theta^A \wedge * d\theta_A * (p_B \wedge \theta^B) - \frac{1}{2} m \wedge \theta^A \wedge * p_A * (d\theta_B \wedge \theta^B) - \\
& - \frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * p_A \wedge * d\theta_B + \frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * d\theta_A \wedge * p_B - \\
& - \frac{1}{2} m \wedge \theta^A \wedge * d\theta_B * (p_A \wedge \theta^B) + \frac{1}{2} m \wedge \theta^A \wedge * p_B * (d\theta_A \wedge \theta^B) + \\
& + m \wedge * (\xi^A p_A) \wedge * (\xi^B d\theta_B) - m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge * p_A). \quad (5.33)
\end{aligned}$$

In order to simplify the expression above we transform its first term:

$$\begin{aligned}
& - * (m \wedge p_B) \theta^B \wedge \theta^A \wedge * d\theta_A = -m \wedge p_B * (\theta^B \wedge \theta^A \wedge * d\theta_A) = \\
& = -m \wedge p_B \vec{\theta}^B \lrcorner * (\theta^A \wedge * d\theta_A) = -\vec{\theta}^B \lrcorner (m \wedge p_B) \wedge * (\theta^A \wedge * d\theta_A) = \\
& = -(\vec{\theta}^B \lrcorner m) p_B \wedge * (\theta^A \wedge * d\theta_A) + m \wedge * (* p_B \wedge \theta^B) \wedge * (\theta^A \wedge * d\theta_A) = \\
& = -(\vec{\theta}^B \lrcorner m) \theta^A \wedge * d\theta_A \wedge * p_B + m \wedge * (\theta^A \wedge * d\theta_A) \wedge * (\theta^B \wedge * p_B).
\end{aligned}$$

Similarly, the second term in (5.33)

$$\begin{aligned}
& * (m \wedge d\theta_B) \theta^B \wedge \theta^A \wedge * p_A = \\
& = (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * p_A \wedge * d\theta_B - m \wedge * (\theta^A \wedge * p_A) \wedge * (\theta^B \wedge * d\theta_B).
\end{aligned}$$

Setting these two expressions to (5.33) we obtain:

$$\begin{aligned}
& \int_{\Sigma} \frac{1}{2} m \wedge \theta^A \wedge * d\theta_A * (p_B \wedge \theta^B) - \frac{1}{2} m \wedge \theta^A \wedge * p_A * (d\theta_B \wedge \theta^B) - \\
& + \frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * p_A \wedge * d\theta_B - \frac{1}{2} (\vec{\theta}^B \lrcorner m) \wedge \theta^A \wedge * d\theta_A \wedge * p_B - \\
& - \frac{1}{2} m \wedge \theta^A \wedge * d\theta_B * (p_A \wedge \theta^B) + \frac{1}{2} m \wedge \theta^A \wedge * p_B * (d\theta_A \wedge \theta^B) + \\
& + m \wedge * (\xi^A p_A) \wedge * (\xi^B d\theta_B) - m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge * p_A) + 2m \wedge * (\theta^A \wedge * d\theta_A) \wedge * (\theta^B \wedge * p_B). \quad (5.34)
\end{aligned}$$

On the other hand by shifting the contraction  $\vec{\theta}^B \lrcorner$  in the third and the fourth terms above we simplify (5.34) to

$$\begin{aligned}
& \int_{\Sigma} (\vec{\theta}^B \lrcorner \theta^A) m \wedge * p_A \wedge * d\theta_B - m \wedge \theta^A \wedge * d\theta_B * (p_A \wedge \theta^B) + m \wedge \theta^A \wedge * p_B * (d\theta_A \wedge \theta^B) + \\
& + m \wedge * (\xi^A p_A) \wedge * (\xi^B d\theta_B) - m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge * p_A) + 2m \wedge * (\theta^A \wedge * d\theta_A) \wedge * (\theta^B \wedge * p_B). \quad (5.35)
\end{aligned}$$



Note that in the expression above there is only one term containing  $\xi^A$  which can be transformed as follows:

$$\begin{aligned}
\int_{\Sigma} m \wedge *(\xi^A p_A) \wedge *(\xi^B d\theta_B) &= \int_{\Sigma} -*[m \wedge *(\xi^B d\theta_B)] \wedge (\xi^A p_A) = -B\left(* (m \wedge \xi^B * d\theta_B)\right) + \\
&+ \int_{\Sigma} *[m \wedge *(\xi^B d\theta_B)] \wedge \theta^A \wedge *d\theta_A = -B\left(* (m \wedge \xi^B * d\theta_B)\right) - \\
- \int_{\Sigma} *[m \wedge *(\theta^A \wedge *d\theta_A)] \wedge \xi^B d\theta_B &= -B\left(* (m \wedge \xi^B * d\theta_B)\right) + R\left(* [m \wedge *(\theta^A \wedge *d\theta_A)]\right) - \\
&- \int_{\Sigma} m \wedge *(\theta^A \wedge *d\theta_A) \wedge *(\theta^B \wedge *p_B). \quad (5.36)
\end{aligned}$$

Gathering the result above, (5.35) and (5.31) we obtain

$$\begin{aligned}
\{S(M), S(M')\} &= V(\vec{m}) + B\left(\theta^B * (m \wedge p_B) - \frac{1}{2}m * (p_B \wedge \theta^B)\right) - \\
&- R\left(\theta^B * (m \wedge d\theta_B) - \frac{1}{2}m * (d\theta_B \wedge \theta^B)\right) - \\
&- B\left(* (m \wedge \xi^B * d\theta_B)\right) + R\left(* [m \wedge *(\theta^A \wedge *d\theta_A)]\right) + \text{remaining terms}, \quad (5.37)
\end{aligned}$$

where the remaining terms read now

$$\begin{aligned}
\int_{\Sigma} (\vec{\theta}^B \lrcorner \theta^A) m \wedge *p_A \wedge *d\theta_B - m \wedge \theta^A \wedge *d\theta_B * (p_A \wedge \theta^B) &+ m \wedge \theta^A \wedge *p_B * (d\theta_A \wedge \theta^B) - \\
- m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge *p_A) &+ m \wedge *(\theta^A \wedge *d\theta_A) \wedge *(\theta^B \wedge *p_B). \quad (5.38)
\end{aligned}$$

Now let us show that the remaining terms (5.38) can be expressed as  $R(b)$  with the one-form  $b$  being a complicated function of the canonical variables and  $m$ . By shifting the contraction  $\vec{\theta}^B \lrcorner$  in the first term above and using (4.9) one can easily show that the sum of the first and the second terms in (5.38) reads

$$\begin{aligned}
\int_{\Sigma} (\vec{\theta}^B \lrcorner \theta^A) m \wedge *p_A \wedge *d\theta_B - m \wedge \theta^A \wedge *d\theta_B * (p_A \wedge \theta^B) &= \\
= \int_{\Sigma} \vec{\theta}^B \lrcorner (m \wedge *d\theta_B) \wedge \theta^A \wedge *p_A &= R\left(\vec{\theta}^B \lrcorner (m \wedge *d\theta_B)\right) + \int_{\Sigma} \vec{\theta}^B \lrcorner (m \wedge *d\theta_B) \wedge \xi^A d\theta_A. \quad (5.39)
\end{aligned}$$

Let us now express the third term in (5.38) by means of the components of the

canonical variables and the covariant derivative  $\nabla_a$ :

$$\begin{aligned} m \wedge \theta^A \wedge *p_B * (d\theta_A \wedge \theta^B) &= m \wedge \frac{1}{2} \theta_i^A p_{Bjk} \epsilon^{jk}_l dx^i \wedge dx^l (\nabla_a \theta_{Ab}) \theta_c^B \epsilon^{abc} = \\ &= m \wedge \theta_i^A p_B^{ab} dx^i \wedge dx^c (\nabla_a \theta_{Ab}) \theta_c^B + m \wedge \theta_i^A p_B^{bc} dx^i \wedge dx^a (\nabla_a \theta_{Ab}) \theta_c^B + \\ &\quad + m \wedge \theta_i^A p_B^{ca} dx^i \wedge dx^b (\nabla_a \theta_{Ab}) \theta_c^B, \end{aligned}$$

where in the last step we used (5.9). Similarly, the fourth term in (5.38)

$$-m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge *p_A) = -m \wedge \theta_i^B \theta_c^A dx^i \wedge dx^c (\nabla_a \theta_{Bb}) p_A^{ab}$$

and consequently the sum of the third and the fourth terms in (5.38) is of the following form:

$$\begin{aligned} m \wedge \theta^A \wedge *p_B * (d\theta_A \wedge \theta^B) - m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge *p_A) &= \\ &= m \wedge \theta_i^A p_B^{bc} dx^i \wedge dx^a (\nabla_a \theta_{Ab}) \theta_c^B + m \wedge \theta_i^A p_B^{ca} dx^i \wedge dx^b (\nabla_a \theta_{Ab}) \theta_c^B = \\ &= m \wedge (d\theta_A)_{ab} p_B^b{}_c \theta^{Bc} \theta_i^A dx^i \wedge dx^a = -m \wedge (\theta^{Bc} p_{Bc}{}^b) (d\theta_A)_{ba} dx^a \wedge \theta^A = \\ &= -m \wedge [(\vec{\theta}^B \lrcorner p_B) \lrcorner d\theta^A] \wedge \theta^A = -m \wedge [*d\theta_A \wedge *(\theta^B \wedge *p_B)] \wedge \theta^A = \\ &= -*[(m \wedge \theta^A) \wedge *d\theta_A] \wedge \theta^B \wedge *p_B \end{aligned}$$

Integrating both sides of the equation above over  $\Sigma$  we obtain:

$$\begin{aligned} \int_{\Sigma} m \wedge \theta^A \wedge *p_B * (d\theta_A \wedge \theta^B) - m \wedge \theta^B \wedge \theta^A * (d\theta_B \wedge *p_A) &= \\ &= R \left( -*[(m \wedge \theta^B) \wedge *d\theta_B] \right) + \int_{\Sigma} -*[(m \wedge \theta^B) \wedge *d\theta_B] \wedge \xi^A d\theta_A. \end{aligned} \quad (5.40)$$

Finally, the last term in (5.38)

$$\begin{aligned} \int_{\Sigma} m \wedge *(\theta^B \wedge *d\theta_B) \wedge *(\theta^A \wedge *p_A) &= \int_{\Sigma} *[m \wedge *(\theta^B \wedge *d\theta_B)] \wedge \theta^A \wedge *p_A = \\ &= R \left( *[m \wedge *(\theta^B \wedge *d\theta_B)] \right) + \int_{\Sigma} *[m \wedge *(\theta^B \wedge *d\theta_B)] \wedge \xi^A d\theta_A. \end{aligned} \quad (5.41)$$

Gathering the three results (5.39), (5.40) and (5.41) we conclude that the terms (5.38) can be expressed as

$$\begin{aligned} R \left( \vec{\theta}^B \lrcorner (m \wedge *d\theta_B) - *[(m \wedge \theta^B) \wedge *d\theta_B] + *[m \wedge *(\theta^B \wedge *d\theta_B)] \right) + \\ + \text{terms independent of } p_A, \end{aligned} \quad (5.42)$$

where the terms independent of  $p_A$  read

$$\int_{\Sigma} \vec{\theta}^B \lrcorner (m \wedge *d\theta_B) \wedge \xi^A d\theta_A - *[(m \wedge \theta^B) \wedge *d\theta_B] \wedge \xi^A d\theta_A + * [m \wedge *(\theta^B \wedge *d\theta_B)] \wedge \xi^A d\theta_A. \quad (5.43)$$

Now it is enough to show that the terms (5.43) sums up to zero for all  $m$  and  $\theta^A$ . To this end let us isolate the factor  $m\xi^A$  in each term of (5.43) obtaining thereby

$$\begin{aligned} \int_{\Sigma} -m\xi^A \wedge *d\theta_B \wedge \vec{\theta}^B \lrcorner d\theta_A - (\vec{\theta}^B \lrcorner *m) \wedge *d\theta_B \wedge \xi^A *d\theta_A - m \wedge \vec{\theta}^B \lrcorner d\theta_B \wedge \xi^A *d\theta_A = \\ = \int_{\Sigma} m\xi^A \wedge [-*d\theta_B \wedge \vec{\theta}^B \lrcorner d\theta_A + *(\vec{\theta}^B \lrcorner (*d\theta_B \wedge *d\theta_A)) - \vec{\theta}^B \lrcorner d\theta_B \wedge *d\theta_A]. \end{aligned}$$

Consider now the terms in the square bracket above:

$$\begin{aligned} -*d\theta_B \wedge \vec{\theta}^B \lrcorner d\theta_A + *(\vec{\theta}^B \lrcorner (*d\theta_B \wedge *d\theta_A)) - \vec{\theta}^B \lrcorner d\theta_B \wedge *d\theta_A = \\ = -*d\theta_B \wedge \vec{\theta}^B \lrcorner d\theta_A + (\vec{\theta}^B \lrcorner *d\theta_B) d\theta_A - d\theta_B \vec{\theta}^B \lrcorner *d\theta_A - \vec{\theta}^B \lrcorner d\theta_B \wedge *d\theta_A = \\ = \vec{\theta}^B \lrcorner (*d\theta_B \wedge d\theta_A) - \vec{\theta}^B \lrcorner (d\theta_B \wedge *d\theta_A) = \vec{\theta}^B \lrcorner (*d\theta_B \wedge d\theta_A) - \vec{\theta}^B \lrcorner (*d\theta_B \wedge d\theta_A) = 0. \end{aligned}$$

This means that indeed (5.43) is zero for all  $m$  and  $\theta^A$ . Consequently,

$$\text{the terms (5.38)} = R \left( \vec{\theta}^B \lrcorner (m \wedge *d\theta_B) - *[(m \wedge \theta^B) \wedge *d\theta_B] + * [m \wedge *(\theta^B \wedge *d\theta_B)] \right). \quad (5.44)$$

Setting this result to (5.37) we obtain

$$\begin{aligned} \{S(M), S(M')\} = V(\vec{m}) + \\ + B \left( \theta^B * (m \wedge p_B) - \frac{1}{2} m * (p_B \wedge \theta^B) \right) - R \left( \theta^B * (m \wedge d\theta_B) - \frac{1}{2} m * (d\theta_B \wedge \theta^B) \right) - \\ - B \left( * (m \wedge \xi^B * d\theta_B) \right) + R \left( * [m \wedge *(\theta^A \wedge *d\theta_A)] \right) + \\ + R \left( \vec{\theta}^B \lrcorner (m \wedge *d\theta_B) - *[(m \wedge \theta^B) \wedge *d\theta_B] + * [m \wedge *(\theta^B \wedge *d\theta_B)] \right) \quad (5.45) \end{aligned}$$

### 5.3.3 Another form of $\{S(M), S(M')\}$

Let us now transform the result to a form in which both constraints  $B$  and  $R$  appear on an equal footing. Consider the following transformations:

$$p_A \mapsto -d\theta_A, \quad d\theta_A \mapsto p_A. \quad (5.46)$$

It is easy to see that under this transformations

$$\begin{aligned} B(a(p_A, d\theta_B)) &\mapsto R(a(-d\theta_A, p_B)), \\ R(b(p_A, d\theta_B)) &\mapsto -B(b(-d\theta_A, p_B)) \end{aligned}$$

Let us now express the formula (5.36) in the following form:

$$\begin{aligned} \int_{\Sigma} m \wedge *(\xi^A p_A) \wedge *(\xi^B d\theta_B) + m \wedge *(\theta^A \wedge *d\theta_A) \wedge *(\theta^B \wedge *p_B) = \\ = -B\left(* (m \wedge \xi^B * d\theta_B)\right) + R\left(* [m \wedge *(\theta^A \wedge *d\theta_A)]\right). \end{aligned}$$

Note now that the l.h.s. of the identity above is invariant with respect to the transformations (5.46). Consequently, the r.h.s. has to be invariant too. Thus the fourth and the fifth terms in (5.45)

$$\begin{aligned} -B\left(* (m \wedge \xi^B * d\theta_B)\right) + R\left(* [m \wedge *(\theta^A \wedge *d\theta_A)]\right) &= -R\left(* (m \wedge \xi^B * p_B)\right) - \\ -B\left(* [m \wedge *(\theta^A \wedge *p_A)]\right) &= -\frac{1}{2}B\left(* (m \wedge \xi^B * d\theta_B) + *[m \wedge *(\theta^A \wedge *p_A)]\right) + \\ + \frac{1}{2}R\left(-*(m \wedge \xi^B * p_B) + *[m \wedge *(\theta^A \wedge *d\theta_A)]\right), \quad (5.47) \end{aligned}$$

where the last equation holds by virtue of the following trivial fact: if  $x = y$  then  $x = \frac{1}{2}(x + y)$ .

Similarly, the expression (5.38) is also invariant under the transformations (5.46). Thus by virtue of the identity (5.44) the last term in (5.45)

$$\begin{aligned} R\left(\vec{\theta}^B \lrcorner (m \wedge *d\theta_B) - *[(m \wedge \theta^B) \wedge *d\theta_B] + *[m \wedge *(\theta^B \wedge *d\theta_B)]\right) = \\ = -B\left(\vec{\theta}^B \lrcorner (m \wedge *p_B) - *[(m \wedge \theta^B) \wedge *p_B] + *[m \wedge *(\theta^B \wedge *p_B)]\right) = \\ = \frac{1}{2}R\left(\vec{\theta}^B \lrcorner (m \wedge *d\theta_B) - *[(m \wedge \theta^B) \wedge *d\theta_B] + *[m \wedge *(\theta^B \wedge *d\theta_B)]\right) - \\ - \frac{1}{2}B\left(\vec{\theta}^B \lrcorner (m \wedge *p_B) - *[(m \wedge \theta^B) \wedge *p_B] + *[m \wedge *(\theta^B \wedge *p_B)]\right). \quad (5.48) \end{aligned}$$

Setting the results (5.47) and (5.48) to (5.45) after some simple calculations we obtain

the final form of the Poisson bracket of the scalar constraints

$$\begin{aligned}
\{S(M), S(M')\} = & V(\vec{m}) + B\left(\theta^B * (m \wedge p_B) - \frac{1}{2} * (m \wedge \xi^B * d\theta_B) - \right. \\
& - * [m \wedge * (\theta^B \wedge * p_B)] - \frac{1}{2} * (*m \wedge \theta^B) * p_B + \frac{1}{2} * [(m \wedge \theta^B) \wedge * p_B] \Big) + \\
& + R\left(-\theta^B * (m \wedge d\theta_B) - \frac{1}{2} * (m \wedge \xi^B * p_B) + \right. \\
& + * [m \wedge * (\theta^B \wedge * d\theta_B)] + \frac{1}{2} * (*m \wedge \theta^B) * d\theta_B - \frac{1}{2} * [(m \wedge \theta^B) \wedge * d\theta_B] \Big) \quad (5.49)
\end{aligned}$$

which coincides with the result shown in Subsection 3.2. Note that now the sum of the constraints  $B$  and  $R$  at the r.h.s. of (5.49) is explicitly invariant under (5.46).

#### 5.4 Poisson bracket of $R(b)$ and $S(M)$

To show that the bracket  $\{R(b), S(M)\}$  is a combination of the constraints known so far we will proceed according to the following prescription. The bracket under consideration can be expressed as

$$\{R(b), S(M)\} = \sum_{i=1}^2 \sum_{j=1}^3 (-1)^{i+1} \{R_i(b), S_j(M)\}.$$

It is not difficult to see that each bracket in the sum is either quadratic in  $p_A$ , linear in  $p_A$  or independent of  $p_A$ . So we will first calculate the brackets and then we will gather the similar terms according to the classification. Next it will turn out that the terms quadratic in  $p_A$  can be reexpressed as a constraint plus a term linear in  $p_A$ . Next, it will turn out that all the linear terms can be reexpressed as some constraints plus a term independent of  $p_A$ . Finally we will show that all the term independent of  $p_A$  sum up to zero.

Except the prescription we will need some formulae and identities which will make easier the calculations.

##### 5.4.1 Useful formulae

The following formulae will be used in the sequel while calculating both  $\{R(b), S(M)\}$  and  $\{B(a), S(M)\}$ :

$$\vec{\theta}_{A \perp} * \xi^B = -\frac{1}{2} \epsilon^B_{CDA} \theta^C \wedge \theta^D + \xi_A * \theta^B, \quad (5.50)$$

$$-\frac{1}{2} \epsilon^A_{BCD} \theta^B \wedge \theta^C \wedge \alpha = -\xi_D \alpha \wedge * \theta^A + (*\xi^A) \vec{\theta}_{D \perp} \alpha, \quad (5.51)$$

$$\begin{aligned}
(\alpha \wedge *_A' \beta) \wedge \gamma &= \alpha \wedge * \beta \vec{\theta}_A \lrcorner \gamma - [(\vec{\theta}_A \lrcorner \alpha) \wedge * \beta + (\alpha \leftrightarrow \beta)] \wedge \gamma = \\
&= [(-1)^k \alpha \wedge (\vec{\theta}_A \lrcorner * \beta) - (\vec{\theta}_A \lrcorner \beta) \wedge * \alpha] \wedge \gamma,
\end{aligned} \tag{5.52}$$

$$\begin{aligned}
(\alpha \wedge *_A' \beta) \wedge \frac{\delta S_1(M)}{\delta p_A} &= M(k - \frac{5}{2}) \alpha \wedge * \beta * (p_A \wedge \theta^A) + \\
&+ (-1)^{4-k} M[* (\alpha \wedge \theta^A) \wedge * p_A \wedge \beta + (\alpha \leftrightarrow \beta)],
\end{aligned} \tag{5.53}$$

$$\begin{aligned}
\alpha^A \wedge \beta_A &= \theta_A \wedge \vec{\theta}^B \lrcorner \alpha^A \wedge \beta_B + (-1)^k \theta_A \wedge \alpha^A \wedge \vec{\theta}^B \lrcorner \beta_B - \\
&- \xi_A \alpha^A \wedge \xi^B \beta_B,
\end{aligned} \tag{5.54}$$

$$d(*(\theta^A \wedge \theta^B)) = \xi^B d\xi^A + \xi^A d\xi^B, \tag{5.55}$$

where in the second formula  $\alpha$  is a one-form, and in the third one  $\alpha$  and  $\beta$  are  $k$ -forms and  $\gamma$  a one-form, and in the fourth one  $\alpha$  and  $\beta$  are  $k$ -forms and finally in the fifth one  $\alpha^A$  is a  $k$ -form, while  $\beta_A$  is a  $(3-k)$ -form. Moreover, we will apply the following two identities:

$$\alpha \wedge *(*\gamma \wedge \beta) - *\alpha *(\beta \wedge \gamma) - \beta \wedge *(*\gamma \wedge \alpha) + *\beta *(\alpha \wedge \gamma) = 0, \tag{5.56}$$

$$*[\alpha \wedge *(\kappa \wedge \beta)] - \alpha *(\beta \wedge *\kappa) - *[\beta \wedge *(\kappa \wedge \alpha)] + \beta *(\alpha \wedge *\kappa) = 0, \tag{5.57}$$

where  $\alpha, \beta$  and  $\kappa$  are one-forms, and  $\gamma$  is a two-form.

Note that if in (5.52)  $\alpha$  and  $\beta$  are three-forms then the formula can be simplified further. Indeed, in this case  $*\beta$  is a zero-form thus

$$(\vec{\theta}_A \lrcorner \alpha) \wedge * \beta \wedge \gamma = \alpha \wedge * \beta \vec{\theta}_A \lrcorner \gamma$$

and

$$(\alpha \wedge *_A' \beta) \wedge \gamma = -\alpha \wedge * \beta \vec{\theta}_A \lrcorner \gamma \tag{5.58}$$

Using this result we can also simplify (5.53)—if  $\alpha$  and  $\beta$  are three-forms then setting to (5.58)  $\gamma = (\delta S_1(M))/(\delta p_A)$  we obtain

$$(\alpha \wedge *_A' \beta) \wedge \frac{\delta S_1(M)}{\delta p_A} = \frac{M}{2} \alpha \wedge * \beta * (p_A \wedge \theta^A). \tag{5.59}$$

*Proof of (5.50).*

$$\begin{aligned}
\vec{\theta}_A \lrcorner * \xi^B &= -\frac{1}{2} \epsilon^B{}_{CDE} (\vec{\theta}_A \lrcorner \theta^C) \theta^D \wedge \theta^E = -\frac{1}{2} \epsilon^B{}_{ADE} \theta^D \wedge \theta^E - \frac{1}{2} \epsilon^B{}_{CDE} \xi_A \xi^C \theta^D \wedge \theta^E = \\
&= -\frac{1}{2} \epsilon^B{}_{CDA} \theta^C \wedge \theta^D + \xi_A * \theta^B,
\end{aligned}$$

where in the second step we used (4.23), and in the last one (5.15).  $\square$

*Proof of (5.51).* By virtue of (4.23) and (5.15) the l.h.s. of (5.51) can be transformed as follows:

$$\begin{aligned} -\frac{1}{2}\epsilon^A{}_{BCD}\theta^B \wedge \theta^C \wedge \alpha &= -\frac{1}{2}\epsilon^A{}_{BCE}\theta^B \wedge \theta^C \wedge (\vec{\theta}_D \lrcorner \theta^E - \xi_D \xi^E)\alpha = \\ &= -\frac{1}{2}\epsilon^A{}_{BCE}\theta^B \wedge \theta^C \wedge (\vec{\theta}_D \lrcorner \theta^E)\alpha - (*\theta^A) \wedge \xi_D \alpha. \end{aligned}$$

Shifting the contraction  $\vec{\theta}_D \lrcorner$  in the first of the two resulting terms and applying once again (4.23) and (5.15) we obtain

$$\begin{aligned} -\frac{1}{2}\epsilon^A{}_{BCD}\theta^B \wedge \theta^C \wedge \alpha &= \epsilon^A{}_{DCE}\theta^C \wedge \theta^E \wedge \alpha - \frac{1}{2}\epsilon^A{}_{BCE}\theta^B \wedge \theta^C \wedge \theta^E (\vec{\theta}_D \lrcorner \alpha) - \\ &\quad - 3(*\theta^A) \wedge \xi_D \alpha. \end{aligned}$$

Now to justify (5.51) it is enough to note that (i) the first term on the r.h.s of the equation above is proportional to the term on the l.h.s. and (ii) the second term on the r.h.s. is proportional to  $(*\xi^A)\vec{\theta}_D \lrcorner \alpha$ .  $\square$

*Proof of (5.52).* Let us now consider the l.h.s. of (5.52):

$$\begin{aligned} (\alpha *'_A \beta) \wedge \gamma &= \vec{\theta}^B \lrcorner \left( \eta_{AB} \alpha \wedge * \beta - [(\vec{\theta}_A \lrcorner \alpha) \wedge *(\vec{\theta}_B \lrcorner \beta) + (\alpha \leftrightarrow \beta)] \right) \wedge \gamma = \\ &= \alpha \wedge * \beta (\vec{\theta}_A \lrcorner \gamma) - [(\vec{\theta}_A \lrcorner \alpha) \wedge *(\vec{\theta}_B \lrcorner \beta) + (\alpha \leftrightarrow \beta)] \vec{\theta}^B \lrcorner \gamma. \end{aligned}$$

By virtue of (4.9) and (4.17)

$$*(\vec{\theta}_B \lrcorner \beta) \vec{\theta}^B \lrcorner \gamma = * \beta \wedge \theta_B \wedge \vec{\theta}^B \lrcorner \gamma = * \beta \wedge \gamma.$$

Setting this result to the r.h.s. of the previous equation we obtain the r.h.s. of the first line of (5.52). To obtain the result at the second line it is enough to shift the contraction  $\vec{\theta}_A \lrcorner$  in the term  $-(\vec{\theta}_A \lrcorner \alpha) \wedge * \beta \wedge \gamma$ .  $\square$

*Proof of (5.53).* By virtue of the first line of Equation (5.52) just proven

$$\begin{aligned} (\alpha \wedge *'_A \beta) \wedge \frac{\delta S_1(M)}{\delta p_A} &= \alpha \wedge * \beta \vec{\theta}_A \lrcorner \frac{\delta S_1(M)}{\delta p_A} - [(\vec{\theta}_A \lrcorner \alpha) \wedge * \beta + (\alpha \leftrightarrow \beta)] \wedge \frac{\delta S_1(M)}{\delta p_A} = \\ &= -\frac{M}{2} \alpha \wedge * \beta * (p_A \wedge \theta^A) - M [(\vec{\theta}_A \lrcorner \alpha) \wedge * \beta + (\alpha \leftrightarrow \beta)] \wedge \left( \theta^B * (p_B \wedge \theta^A) - \frac{1}{2} \theta^A * (p_B \wedge \theta^B) \right). \end{aligned} \tag{5.60}$$

Because  $*(p_B \wedge \theta^A)$  is a zero-form (i.e. a function)

$$(\vec{\theta}_A \lrcorner \alpha) * (p_B \wedge \theta^A) = *(*\alpha \wedge \theta_A) * (p_B \wedge \theta^A) = *(*\alpha \wedge \theta_A \wedge *(p_B \wedge \theta^A)) = *(*\alpha \wedge *p_B),$$

where the first equality holds true by virtue of (4.9) and the last one—due to (5.13). On the other hand due to (4.17)

$$(\vec{\theta}_A \lrcorner \alpha) \wedge * \beta \wedge \theta^A = \theta^A \wedge (\vec{\theta}_A \lrcorner \alpha) \wedge * \beta = k \alpha \wedge * \beta.$$

Setting the two results above to (5.60) we get

$$\begin{aligned} (\alpha \wedge *_A' \beta) \wedge \frac{\delta S_1(M)}{\delta p_A} &= M(k - \frac{1}{2}) \alpha \wedge * \beta * (p_A \wedge \theta^A) - \\ &\quad - M[*(*\alpha \wedge *p_A) \wedge * \beta \wedge \theta^A + (\alpha \leftrightarrow \beta)]. \end{aligned} \quad (5.61)$$

The last step of the proof aims at simplifying the last term in the equation above:

$$\begin{aligned} &*(*\alpha \wedge *p_A) \wedge * \beta \wedge \theta^A = *\alpha \wedge *p_A \wedge *(*\beta \wedge \theta^A) = *\alpha \wedge *p_A \wedge \vec{\theta}^A \lrcorner \beta = \\ &= (-1)^{3-k} \vec{\theta}^A \lrcorner (*\alpha \wedge *p_A) \wedge \beta = (-1)^{3-k} *(\alpha \wedge \theta^A) \wedge *p_A \wedge \beta + *\alpha \wedge \beta * (p_A \wedge \theta^A) \end{aligned}$$

(note that here we used (4.9) several times). Taking into account that  $*\alpha \wedge \beta = \alpha \wedge *\beta$  we set the result above to (5.61) obtaining thereby (5.53).  $\square$

*Proof of (5.54).* By virtue of (4.23)

$$\alpha^A \wedge \beta_A = (\vec{\theta}^B \lrcorner \theta_A - \xi^B \xi_A) \alpha^A \wedge \beta_B = \vec{\theta}^B \lrcorner \theta_A \alpha^A \wedge \beta_B - \xi_A \alpha^A \wedge \xi^B \beta_B.$$

Now to get (5.54) it is enough to shift the contraction  $\vec{\theta}^B \lrcorner$  in the first term at the r.h.s. of the equation above.  $\square$

*Proof of (5.55).* First transform the l.h.s. of (4.23) by means of (4.9) then act on the both sides of the resulting formula by  $d$ .  $\square$

*Proof of (5.56).* Note that  $\beta \wedge \gamma$  is a three form. Therefore

$$*\alpha * (\beta \wedge \gamma) = *[* (\beta \wedge \gamma) \wedge \alpha] = \vec{\alpha} \lrcorner (\beta \wedge \gamma) = \vec{\alpha} \lrcorner \beta \wedge \gamma - \beta \wedge \vec{\alpha} \lrcorner \gamma,$$

where in the second step we used (4.9). Transforming similarly the term  $*\beta * (\alpha \wedge \gamma)$  we obtain

$$\begin{aligned} \alpha \wedge * (*\gamma \wedge \beta) - *\alpha * (\beta \wedge \gamma) - \beta \wedge * (*\gamma \wedge \alpha) + *\beta * (\alpha \wedge \gamma) &= \\ &= \alpha \wedge \vec{\beta} \lrcorner \gamma - \vec{\alpha} \lrcorner \beta \wedge \gamma + \beta \wedge \vec{\alpha} \lrcorner \gamma - \beta \wedge \vec{\alpha} \lrcorner \gamma + \vec{\beta} \lrcorner \alpha \wedge \gamma - \alpha \wedge \vec{\beta} \lrcorner \gamma = 0. \end{aligned}$$

$\square$

*Proof of (5.57).* Act by the star  $*$  on both sides of (5.56) and set  $\kappa = *\gamma$ .  $\square$

Now we are ready to begin the calculations of  $\{R(B), S(M)\}$ .



### 5.4.2 Terms quadratic in $p_A$

Terms quadratic in the momentum come from the Poisson bracket

$$\begin{aligned} \{R_1(b), S_1(M)\} &= \int_{\Sigma} [-b \wedge *p_A + (b \wedge \theta^B) \wedge *_A' p_B] \wedge \frac{\delta S_1(M)}{\delta p_A} - \\ &\quad - M \left( p_B * (p_A \wedge \theta^B) - \frac{1}{2} p_A * (p_B \wedge \theta^B) \right) \wedge *(b \wedge \theta^A), \end{aligned} \quad (5.62)$$

where we used (5.12) to simplify the r.h.s. It is not difficult to see that

$$-b \wedge *p_A \wedge \frac{\delta S_1(M)}{\delta p_A} - M \left( p_B * (p_A \wedge \theta^B) - \frac{1}{2} p_A * (p_B \wedge \theta^B) \right) \wedge *(b \wedge \theta^A) = 0.$$

Let us now transform the remaining term in (5.62)—by virtue of (5.53)

$$\begin{aligned} [(b \wedge \theta^B) \wedge *_A' p_B] \wedge \frac{\delta S_1(M)}{\delta p_A} &= -\frac{M}{2} b \wedge \theta^B \wedge *p_B * (p_A \wedge \theta^A) - \\ &\quad - M [* (b \wedge \theta^B \wedge \theta^A) \wedge *p_A \wedge p_B + *(p_B \wedge \theta^A) \wedge *p_A \wedge b \wedge \theta^B] = \\ &= -\frac{M}{2} b \wedge \theta^B \wedge *p_B * (p_A \wedge \theta^A) + M * (p_B \wedge \theta^A) * p_A \wedge b \wedge \theta^B. \end{aligned}$$

To justify the last step let us note that  $(b \wedge \theta^B \wedge \theta^A)$  is antisymmetric in  $AB$  while  $*p_A \wedge p_B$  is symmetric. Consequently,

$$\{R_1(b), S_1(M)\} = \int_{\Sigma} -\frac{M}{2} b \wedge \theta^B \wedge *p_B * (p_A \wedge \theta^A) + M * (p_B \wedge \theta^A) b \wedge \theta^B \wedge *p_A. \quad (5.63)$$

### 5.4.3 Terms linear in $p_A$

Here we will calculate the brackets  $\{R_1(b), S_2(M)\}$  and  $\{R_2(b), S_1(M)\}$ , which give terms linear in the momentum.

Considering  $\{R_1(b), S_2(M)\}$  we immediately see that by virtue of (5.12) and (5.14)

$$\frac{\delta S_2(M)}{\delta \theta^A} \wedge \frac{\delta R_1(b)}{\delta p_A} = 0,$$

thus

$$\begin{aligned} \{R_1(b), S_2(M)\} &= \int_{\Sigma} [-b \wedge *p_A + (b \wedge \theta^B) \wedge *_A' p_B] \wedge d(M\xi^A) = \int_{\Sigma} -dM \wedge b \wedge *(\xi^A p_A) - \\ &\quad - Mb \wedge *p_A \wedge d\xi^A + Mb \wedge \theta^B \wedge d\xi^A \vec{\theta}_{A\perp} * p_B - M \vec{\theta}_{A\perp} p_B \wedge *(b \wedge \theta^B) \wedge d\xi^A, \end{aligned} \quad (5.64)$$

where we have used (5.52).

The other bracket,

$$\{R_2(b), S_1(M)\} = \int_{\Sigma} [d(b\xi_A) + (b \wedge d\theta^B) \wedge *_A' (*\xi_B) - \frac{1}{2}\epsilon_{DBCA}\theta^B \wedge \theta^C * (b \wedge d\theta^D)] \wedge \frac{\delta S_1(M)}{\delta p_A}. \quad (5.65)$$

The last two terms in the square bracket above give together zero once multiplied by  $(\delta S_1(M))/(\delta p_A)$ . Indeed, due to (5.59)

$$[(b \wedge d\theta^B) \wedge *_A' (*\xi_B)] \wedge \frac{\delta S_1(M)}{\delta p_A} = \frac{M}{2}(b \wedge \xi_B d\theta^B) * (p_A \wedge \theta^A).$$

On the other hand, by virtue of (5.51)

$$\begin{aligned} -\frac{1}{2}\epsilon_{DBCA}\theta^B \wedge \theta^C \wedge \frac{\delta S_1(M)}{\delta p_A} * (b \wedge d\theta^D) &= *\xi_D * (b \wedge d\theta^D) \vec{\theta}_{A\lrcorner} \frac{\delta S_1(M)}{\delta p_A} = \\ &= -\frac{M}{2}(b \wedge \xi_B d\theta^B) * (p_A \wedge \theta^A) \end{aligned}$$

—observe that  $\xi_A(\delta S_1(M))/(\delta p_A) = 0$  because  $(\delta S_1(M))/(\delta p_A)$  is of the form  $\theta_i^A \gamma^i_j dx^j$  form some tensor field  $\gamma^i_j$ . Using this observation we transform the only remaining term in (5.65) as follows:

$$\begin{aligned} d(b\xi_A) \wedge \frac{\delta S_1(M)}{\delta p_A} &= -Mb \wedge d\xi_A \wedge \theta^B * (p_B \wedge \theta^A) + \frac{M}{2}b \wedge d\xi_A \wedge \theta^A * (p_B \wedge \theta^B) = \\ &= Mb \wedge \theta^B \wedge d\xi^A \vec{\theta}_{A\lrcorner} * p_B - \frac{M}{2}b \wedge \theta^A \wedge d\xi_A * (p_B \wedge \theta^B). \quad (5.66) \end{aligned}$$

Gathering all the terms linear in  $p_A$ , that is, (5.64) and (5.66) we obtain

$$\begin{aligned} \{R_1(b), S_2(M)\} - \{R_2(b), S_1(M)\} &= \int_{\Sigma} -dM \wedge b \wedge *(\xi^A p_A) - Mb \wedge *p_A \wedge d\xi^A - \\ &\quad - M\vec{\theta}_{A\lrcorner} p_B \wedge *(b \wedge \theta^B) \wedge d\xi^A + \frac{M}{2}b \wedge \theta^A \wedge d\xi_A * (p_B \wedge \theta^B). \quad (5.67) \end{aligned}$$

#### 5.4.4 Terms independent of $p_A$

It turns out that the remaining three brackets,  $\{R_2(b), S_3(M)\}$ ,  $\{R_2(b), S_2(M)\}$  and  $\{R_1(b), S_3(M)\}$  do not depend on the momentum. The first bracket is zero since both  $R_2(b)$  and  $S_3(M)$  do not contain  $p_A$ . The second bracket by virtue of (5.14) and (5.12) reads

$$\{R_2(b), S_2(M)\} = \int_{\Sigma} [(b \wedge d\theta^B) \wedge *_A' (*\xi_B) - \frac{1}{2}\epsilon_{DBCA}\theta^B \wedge \theta^C * (b \wedge d\theta^D)] \wedge d(M\xi^A).$$

Applying (5.58), (5.15) and (5.51) it is not difficult to show that

$$\{R_2(b), S_2(M)\} = \int_{\Sigma} dM \wedge * \theta_D * (b \wedge d\theta^D).$$

Consider now the last bracket  $\{R_1(b), S_3(M)\}$ . Due to (5.12)

$$\begin{aligned} \{R_1(b), S_3(M)\} = - \int_{\Sigma} & \left( d[M\theta^B * (d\theta_B \wedge \theta_A) - \frac{M}{2} \theta_A * (d\theta_B \wedge \theta^B)] + M d\theta_B * (d\theta_A \wedge \theta^B) - \right. \\ & \left. - \frac{M}{2} d\theta_A * (d\theta_B \wedge \theta^B) \right) \wedge * (b \wedge \theta^A). \end{aligned}$$

Our goal now is to express the r.h.s. of the equation above as a sum of a term containing  $Mb$  and a one containing  $dM$ . To this end we first act by the operator  $d$  on the factors constituting terms in the square brackets. Next, in those cases when it is possible, we use (5.13) to simplify  $\theta_A \wedge * (b \wedge \theta^A)$  to  $2 * b$  and  $\theta_A \wedge * (d\theta_B \wedge \theta^A)$  to  $*d\theta_B$ , finally in all the terms containing  $M$  and  $*b$  we shift the  $*$  to obtain  $b$ . Thus we obtain

$$\begin{aligned} \{R_1(b), S_3(M)\} = - \int_{\Sigma} & Mb \wedge [- * d\theta^A \wedge * d\theta_A + \theta^A \wedge * d\theta^B * (d\theta_A \wedge \theta_B) - \\ & - \theta^A \wedge * d\theta_A * (d\theta_B \wedge \theta^B) - \theta^A \wedge * (\theta^B \wedge d * (d\theta_B \wedge \theta_A)) - * d * (d\theta_A \wedge \theta^A)] + \\ & + dM \wedge [\theta^A \wedge * (b \wedge * d\theta_A) - * b * (d\theta_A \wedge \theta^A)]. \end{aligned}$$

Note that since  $*d\theta^A$  is a one-form the first term at the r.h.s. above vanishes. Thus the terms independent of  $p_A$  read

$$\begin{aligned} \{R_1(b), S_3(M)\} - \{R_2(b), S_2(M)\} = \int_{\Sigma} & \text{the term containing } Mb - \\ & - \int_{\Sigma} dM \wedge [\theta^B \wedge * (b \wedge * d\theta_B) - * b * (d\theta_B \wedge \theta^B) + * \theta_D * (b \wedge d\theta^D)]. \end{aligned}$$

The term containing  $dM$  can be expressed as

$$- \int_{\Sigma} dM \wedge [- * b * (\theta^B \wedge d\theta_B) - \theta^B \wedge * (*d\theta_B \wedge b) + * \theta^B * (b \wedge d\theta_B)].$$

Now by setting in (5.56)  $\alpha = b$ ,  $\beta = \theta^B$  and  $\gamma = d\theta_B$  the term under consideration can be simplified to

$$- \int_{\Sigma} dM \wedge [-b \wedge * (*d\theta_B \wedge \theta^B)] = - \int_{\Sigma} * (dM \wedge b) \wedge \theta^B \wedge * d\theta_B.$$

This means that the terms independent of  $p_A$  read

$$\begin{aligned} \{R_1(b), S_3(M)\} - \{R_2(b), S_2(M)\} = & - \int_{\Sigma} Mb \wedge [\theta^A \wedge *d\theta^B * (d\theta_A \wedge \theta_B) - \\ & - \theta^A \wedge *d\theta_A * (d\theta_B \wedge \theta^B) - \theta^A \wedge *(\theta^B \wedge d * (d\theta_B \wedge \theta_A)) - *d * (d\theta_A \wedge \theta^A)] - \\ & - \int_{\Sigma} *(dM \wedge b) \wedge \theta^A \wedge *d\theta_A. \end{aligned} \quad (5.68)$$

#### 5.4.5 Isolating constraints

Our goal now is to express the bracket

$$\begin{aligned} \{R(b), S(M)\} = & \text{the terms (5.63) quadratic in } p_A + \\ & + \text{the terms (5.67) linear in } p_A + \text{the terms (5.68) independent of } p_A \end{aligned}$$

as a combination of the constraints.

**Terms quadratic in  $p_A$**  We are going to transform the last term of (5.63) to a form containing the factor  $\theta^A \wedge *p_A$  being a part of the constraint  $R(b)$ : applying (5.54) to the term with  $\alpha^A = *p^A$  we obtain

$$\begin{aligned} & \left( M * (p_B \wedge \theta^A) \wedge b \wedge \theta^B \right) \wedge *p_A = *p^A \wedge \left( M * (p_C \wedge \theta_A) \wedge b \wedge \theta^C \right) = \\ & = \theta_A \wedge \tilde{\theta}^B \lrcorner *p^A \wedge \left( M * (p_C \wedge \theta_B) \wedge b \wedge \theta^C \right) - \theta_A \wedge *p^A \wedge \tilde{\theta}^B \lrcorner \left( M * (p_C \wedge \theta_B) \wedge b \wedge \theta^C \right). \end{aligned} \quad (5.69)$$

The first term in the last line is zero—indeed, using (4.9) we get

$$\theta_A \wedge \tilde{\theta}^B \lrcorner *p^A \wedge \left( M * (p_C \wedge \theta_B) \wedge b \wedge \theta^C \right) = -M * (p_A \wedge \theta^B) * (p_C \wedge \theta_B) \theta^A \wedge \theta^C \wedge b.$$

Note now that  $*(p_A \wedge \theta^B) * (p_C \wedge \theta_B)$  is symmetric in  $AC$ , while  $\theta^A \wedge \theta^C$  antisymmetric. Transforming the remaining term in (5.69) we obtain

$$\begin{aligned} & \left( M * (p_B \wedge \theta^A) \wedge b \wedge \theta^B \right) \wedge *p_A = \\ & = -M \theta_A \wedge *p^A \wedge \left( * (p_C \wedge \theta_B \tilde{\theta}^B \lrcorner b) \wedge \theta^C - * (p_C \wedge \theta^B) b \tilde{\theta}^B \lrcorner \theta^C \right) = \\ & = -M \theta^A \wedge *p_A \left( * (p_C \wedge b) \theta^C - * (p_B \wedge \theta^B) \wedge b \right). \end{aligned}$$

where in the last step we used (4.17) and (4.23).

Setting this result to (5.63) we arrive at

$$\begin{aligned} \text{the terms (5.63) quadratic in } p_A &= - \int_{\Sigma} M[\theta^B * (p_B \wedge b) - \frac{1}{2} b * (p_B \wedge \theta^B)] \wedge \theta^A \wedge * p_A = \\ &= -R \left( M[\theta^B * (p_B \wedge b) - \frac{1}{2} b * (p_B \wedge \theta^B)] \right) - \int_{\Sigma} M[\theta^B * (p_B \wedge b) - \frac{1}{2} b * (p_B \wedge \theta^B)] \wedge \theta^A \wedge d\xi_A. \end{aligned} \quad (5.70)$$

Consequently, the bracket  $\{R(b), S(M)\}$  is of the following form

$$\begin{aligned} \{R(b), S(M)\} &= -R \left( M[\theta^B * (p_B \wedge b) - \frac{1}{2} b * (p_B \wedge \theta^B)] \right) + \text{terms linear in } p_A + \\ &\quad + \text{terms independent of } p_A, \end{aligned} \quad (5.71)$$

where now the expression “terms linear in  $p_A$ ” means the terms (5.67) and the last term in (5.70). Now we are going to isolate constraints from the linear terms.

**Terms linear in  $p_A$**  The terms under consideration read

$$\begin{aligned} \int_{\Sigma} -dM \wedge b \wedge *(\xi^A p_A) + Mb \wedge d\xi^A \wedge *p_A - M\vec{\theta}_A \lrcorner p_B \wedge *(b \wedge \theta^B) \wedge d\xi^A + \\ + Mb \wedge \theta^A \wedge d\xi^A * (p_B \wedge \theta^B) - M\theta^B \wedge \theta^A \wedge d\xi_A * (p_B \wedge b). \end{aligned} \quad (5.72)$$

The first term can be written as

$$\int_{\Sigma} -dM \wedge b \wedge *(\xi^A p_A) = \int_{\Sigma} -(dM \wedge b) \wedge \xi^A p_A = -B(* (dM \wedge b)) + \int_{\Sigma} *(dM \wedge b) \wedge \theta^A \wedge *d\theta_A. \quad (5.73)$$

Transformation of the remaining terms in (5.72) (i.e. those which do not contain  $dM$ ) to an appropriate form takes more effort. Applying (5.54) to the first of them by setting  $\beta_A = *p_A$  we obtain

$$\begin{aligned} Mb \wedge d\xi^A \wedge *p_A &= M\theta_A \wedge \vec{\theta}^B \lrcorner (b \wedge d\xi^A) \wedge *p_B + M\theta_A \wedge b \wedge d\xi^A \vec{\theta}^B \lrcorner *p_B = \\ &= M\theta_A \wedge d\xi^A \wedge *p_B \vec{\theta}^B \lrcorner b - M\theta_A \wedge b \wedge *p_B \vec{\theta}^B \lrcorner d\xi^A - Mb \wedge \theta_A \wedge d\xi^A * (p_B \wedge \theta^B). \end{aligned}$$

The last term of (5.72) can be transformed as follows

$$\begin{aligned} -M\theta^B \wedge \theta^A \wedge d\xi_A * (p_B \wedge b) &= -M\theta^A \wedge d\xi_A \wedge [* (p_B \wedge b) \wedge \theta^B] = \\ &= -M\theta^A \wedge d\xi_A \wedge *[\vec{\theta}^B \lrcorner (p_B \wedge b)] = -M\theta^A \wedge d\xi_A \wedge *[(\vec{\theta}^B \lrcorner p_B) \wedge b] - M\theta^A \wedge d\xi_A \wedge *p_B \vec{\theta}^B \lrcorner b, \end{aligned}$$

where in the second step we used (4.9). The two last results allow us to express in a more simpler form the sum of the terms in (5.72) which do not contain  $dM$ :

$$-M\theta_A \wedge b \wedge *p_B \vec{\theta}^B \lrcorner d\xi^A - M\vec{\theta}_A \lrcorner p_B \wedge *(b \wedge \theta^B) \wedge d\xi^A - M\theta^A \wedge d\xi_A \wedge *[(\vec{\theta}^B \lrcorner p_B) \wedge b]. \quad (5.74)$$

Our goal now is to rewrite the sum above in a form of a single term containing the factor  $\theta^A \wedge *p_A$ . Let us transform the first term in (5.74) by means of (5.54) setting  $\beta_A = p_A$ :

$$\begin{aligned} -M\theta_A \wedge b \wedge *p_B \vec{\theta}^B \lrcorner d\xi^A &= M \left( * (b \wedge \theta_C) \vec{\theta}^A \lrcorner d\xi^C \right) \wedge p_A = \\ &= M\theta_A \wedge \vec{\theta}_B \lrcorner \left( * (b \wedge \theta_C) \vec{\theta}^A \lrcorner d\xi^C \right) \wedge p_B - M\theta_A \left( * (b \wedge \theta_C) \vec{\theta}^A \lrcorner d\xi^C \right) \wedge \vec{\theta}^B \lrcorner p_B = \\ &= M\theta_A (\vec{\theta}^A \lrcorner d\xi^C) \wedge *(b \wedge \theta_C \wedge \theta^B) \wedge p_B - M\theta_A (\vec{\theta}^A \lrcorner d\xi^C) \wedge *(b \wedge \theta_C) \wedge *( *p_B \wedge \theta^B) = \\ &= Md\xi^C \wedge *(b \wedge \theta_C \wedge \theta^B) \wedge p_B + M * [d\xi^C \wedge *(b \wedge \theta_C)] \wedge \theta^B \wedge *p_B, \end{aligned} \quad (5.75)$$

where we applied (4.9) in the third step and (4.17) in the last step. The second term in (5.74)

$$\begin{aligned} -M\vec{\theta}_A \lrcorner p_B \wedge *(b \wedge \theta^B) \wedge d\xi^A &= Mp_B [\vec{\theta}_A \lrcorner *(b \wedge \theta^B)] \wedge d\xi^A - Mp_B \wedge *(b \wedge \theta^B) \vec{\theta}_A \lrcorner d\xi^A = \\ &= Mp_B * (b \wedge \theta^B \wedge \theta_A) \wedge d\xi^A - M\vec{\theta}_C \lrcorner d\xi^C * (b \wedge \theta^A) \wedge p_A. \end{aligned}$$

Now we apply (5.54) to the last term in the equation above by setting  $\beta_A = p_A$  and obtain thereby

$$\begin{aligned} -M\vec{\theta}_A \lrcorner p_B \wedge *(b \wedge \theta^B) \wedge d\xi^A &= Md\xi^C * (b \wedge \theta^B \wedge \theta_C) \wedge p_B - \\ &\quad - M\vec{\theta}_C \lrcorner d\xi^C \theta_A [\vec{\theta}^B \lrcorner *(b \wedge \theta^A)] \wedge p_B + M\vec{\theta}_C \lrcorner d\xi^C \theta_A \wedge *(b \wedge \theta^A) \wedge \vec{\theta}^B \lrcorner p_B = \\ &= -Md\xi^C * (b \wedge \theta_C \wedge \theta^B) \wedge p_B + M\vec{\theta}_C \lrcorner d\xi^C \theta_A * (b \wedge \theta^B \wedge \theta^A) \wedge p_B + 2M\vec{\theta}_C \lrcorner d\xi^C * b \wedge \vec{\theta}^B \lrcorner p_B = \\ &= -Md\xi^C * (b \wedge \theta_C \wedge \theta^B) \wedge p_B + M\vec{\theta}_C \lrcorner d\xi^C * (b \wedge \theta^B) \wedge p_B - 2M\vec{\theta}_C \lrcorner d\xi^C b \wedge \theta^B \wedge *p_B = \\ &= -Md\xi^C * (b \wedge \theta_C \wedge \theta^B) \wedge p_B - M\vec{\theta}_C \lrcorner d\xi^C b \wedge \theta^B \wedge *p_B, \end{aligned} \quad (5.76)$$

where we used (4.9) and (5.13). Finally, the last term in (5.74) can be written as

$$\begin{aligned} -M\theta^A \wedge d\xi_A \wedge *[(\vec{\theta}^B \lrcorner p_B) \wedge b] &= -M * (*p_B \wedge \theta^B) \wedge b \wedge *(\theta^A \wedge d\xi_A) = \\ &= M * [b \wedge *(\theta^A \wedge d\xi_A)] \wedge \theta^B \wedge *p_B. \end{aligned}$$

Gathering the three partial results, that is, (5.75), (5.76) and the equation above we obtain the desired expression for those terms in (5.72) which do not contain  $dM$ :

$$\begin{aligned} \int_{\Sigma} M \left( * [d\xi^C \wedge *(b \wedge \theta_C)] - \vec{\theta}_C \lrcorner d\xi^C b + * [b \wedge *(\theta^A \wedge d\xi_A)] \right) \wedge \theta^B \wedge *p_B = \\ = \int_{\Sigma} M \left( * [b \wedge *(\theta^A \wedge d\xi_A)] - b * (d\xi_A \wedge * \theta^A) - * [d\xi_A \wedge *(\theta^A \wedge b)] \right) \wedge \theta^B \wedge *p_B. \end{aligned}$$

We can now simplify the term in the big parenthesis — it is enough to use (5.57) setting  $\alpha = b$ ,  $\beta = d\xi_A$  and  $\kappa = \theta^A$  to get

$$\begin{aligned} \int_{\Sigma} M \left( -d\xi_A * (b \wedge * \theta^A) \right) \wedge \theta^B \wedge * p_B &= -R \left( M d\xi_A * (b \wedge * \theta^A) \right) - \\ &\quad - \int_{\Sigma} M * (b \wedge * \theta^A) d\xi_A \wedge \xi^B d\theta_B, \end{aligned} \quad (5.77)$$

which is the final form of the terms in (5.72) which do not contain  $dM$ .

The equations (5.73) and (5.77) allow us to express the terms linear in  $p_A$  appearing in (5.71) as a sum of constraints and terms independent of the momentum. Consequently, (5.71) can be written now in the following form:

$$\begin{aligned} \{R(b), S(M)\} &= -R \left( M [\theta^B * (p_B \wedge b) - \frac{1}{2} b * (p_B \wedge \theta^B) + d\xi_A * (b \wedge * \theta^A)] \right) - B \left( * (dM \wedge b) \right) + \\ &\quad + \text{terms independent of } p_A, \end{aligned} \quad (5.78)$$

where “terms independent of  $p_A$ ” means here the terms given by (5.68), the last term in (5.73) and the last one in (5.77).

**Terms independent of  $p_A$**  Gathering all the terms independent of  $p_A$  appearing in (5.78) we see that the last term of (5.68) and the last term of (5.73) sum up to zero. Note now that the last term in (5.77) contains  $\xi^A$  which does not appear in the others term. To get rid of it let us use (5.55):

$$\begin{aligned} - \int_{\Sigma} M * (b \wedge * \theta^A) d\xi_A \wedge \xi^B d\theta_B &= - \int_{\Sigma} M * (b \wedge * \theta^A) d * (* \theta_A \wedge \theta^B) \wedge d\theta_B = \\ &= - \int_{\Sigma} M b \wedge * \theta^A * [d * (* \theta_A \wedge \theta^B) \wedge d\theta_B]. \end{aligned}$$

Consequently, the terms in (5.78) independent of  $p_A$  read

$$\begin{aligned} - \int_{\Sigma} M b \wedge \left( \theta^A \wedge * d\theta^B * (d\theta_A \wedge \theta_B) - \theta^A \wedge * d\theta_A * (d\theta_B \wedge \theta^B) - \theta^A \wedge * (\theta^B \wedge d * (d\theta_B \wedge \theta_A)) - \right. \\ \left. - * d * (d\theta_A \wedge \theta^A) + * \theta^A * (d * (* \theta_A \wedge \theta^B) \wedge d\theta_B) \right). \end{aligned} \quad (5.79)$$

Now we are going to show that the expression above is zero for every  $M, b$  and  $\theta^A$ . This will be achieved by proving that the terms in the big parenthesis sum up to zero for every  $\theta^A$ . The proof will be carried out with applications of tensor calculus.

The first term in (5.79)

$$\theta^A \wedge * d\theta^B * (d\theta_A \wedge \theta_B) = (\nabla_i \theta_{Aj}) (\nabla^a \theta^{Bb}) \theta_{Bk} \theta_d^A \epsilon_{abc} \epsilon^{ijk} dx^d \wedge dx^c.$$

By virtue of the third equation in (5.9) we can express the r.h.s. above as a sum of six terms. Four of them vanish: two of them contain the vanishing factor  $(\nabla_a \theta^{Bb})\theta_{Bb}$  (see (5.5)). The other two vanish because they are of the form

$$\gamma_{dc} dx^d \wedge dx^c \quad \text{with} \quad \gamma_{dc} = \gamma_{cd}. \quad (5.80)$$

Thus

$$\begin{aligned} \theta^A \wedge *d\theta^B * (d\theta_A \wedge \theta_B) &= (\nabla_b \theta_{Ac})(\nabla^a \theta^{Bb})\theta_{Ba}\theta_d^A dx^d \wedge dx^c - \\ &\quad - (\nabla_c \theta_{Ab})(\nabla^a \theta^{Bb})\theta_{Ba}\theta_d^A dx^d \wedge dx^c. \end{aligned} \quad (5.81)$$

In the case of the second term in (5.79) we proceed similarly—after applying (5.9) we obtain six terms and again four of them vanish: two of them are of the form (5.80), the other two can be transformed to this form by means of (5.4) and we obtain

$$\begin{aligned} -\theta^A \wedge *d\theta_A * (d\theta_B \wedge \theta^B) &= -(\nabla^a \theta_A^b)(\nabla_c \theta_a^B)\theta_{Bb}\theta_d^A dx^d \wedge dx^c + \\ &\quad + (\nabla^a \theta_A^b)(\nabla_c \theta_b^B)\theta_{Ba}\theta_d^A dx^d \wedge dx^c. \end{aligned} \quad (5.82)$$

Let us consider now the third term in (5.79):

$$\begin{aligned} -\theta^A \wedge *(\theta^B \wedge d * (d\theta_B \wedge \theta_A)) &= -\nabla^b [(\nabla_i \theta_{Bj})\theta_{Ak}]\theta^{Ba}\theta_d^A \epsilon_{abc} \epsilon^{ijk} dx^d \wedge dx^c = \\ &= -(\nabla^b \nabla_i \theta_{Bj})\theta^{Ba} q_{kd} \epsilon_{abc} \epsilon^{ijk} dx^d \wedge dx^c - (\nabla_i \theta_{Bj})(\nabla^b \theta_{Ak})\theta^{Ba}\theta_d^A \epsilon_{abc} \epsilon^{ijk} dx^d \wedge dx^c. \end{aligned}$$

Applying (5.9) we obtain twelve terms: six of them contain a second derivative of components of  $\theta^B$ , while the remaining ones are quadratic in first derivatives of the components. Three terms of those containing the second derivatives vanish: two terms turn out to be of the form (5.80), the third one can be transformed to this form as follows

$$\begin{aligned} -(\nabla_d \nabla_c \theta_{Ba})\theta^{Ba} dx^d \wedge dx^c &= -\nabla_d [(\nabla_c \theta_{Ba})\theta^{Ba}] + (\nabla_c \theta_{Ba})(\nabla_d \theta^{Ba}) dx^d \wedge dx^c = \\ &= (\nabla_c \theta_{Ba})(\nabla_d \theta^{Ba}) dx^d \wedge dx^c. \end{aligned}$$

Four terms of those quadratic in the first derivatives are zero: two of them contain the factor (5.5), the other two can be transformed to the form (5.80) by means of (5.4). Finally

$$\begin{aligned} -\theta^A \wedge *(\theta^B \wedge d * (d\theta_B \wedge \theta_A)) &= -(\nabla^b \nabla_b \theta_{Bc})\theta_d^B dx^d \wedge dx^c + (\nabla_d \nabla_a \theta_{Bc})\theta^{Ba} dx^d \wedge dx^c + \\ &\quad + (\nabla^b \nabla_c \theta_{Bb})\theta_d^B dx^d \wedge dx^c - (\nabla_a \theta_{Bb})(\nabla^b \theta_{Ac})\theta^{Ba}\theta_d^A dx^d \wedge dx^c + \\ &\quad + (\nabla_c \theta_{Bb})(\nabla^b \theta_{Aa})\theta^{Ba}\theta_d^A dx^d \wedge dx^c. \end{aligned} \quad (5.83)$$



Since

$$d\theta_A \wedge \theta^A = 3!(\nabla_{[d}\theta_{Ac})\theta_b^A dx^d \otimes dx^c \otimes dx^b$$

we can use (5.8) to express the fourth term in (5.79) as follows

$$\begin{aligned} - *d * (d\theta_A \wedge \theta^A) &= -\frac{3!}{2}\nabla^b[(\nabla_{[d}\theta_{Ac})\theta_b^A] dx^d \wedge dx^c = -\nabla^b[(\nabla_d\theta_{Ac})\theta_b^A] dx^d \wedge dx^c - \\ &\quad - \nabla^b[(\nabla_c\theta_{Ab})\theta_d^A] dx^d \wedge dx^c - \nabla^b[(\nabla_b\theta_{Ad})\theta_c^A] dx^d \wedge dx^c. \end{aligned}$$

Acting by  $\nabla^b$  on the factors in the square brackets we obtain

$$\begin{aligned} - *d * (d\theta_A \wedge \theta^A) &= -(\nabla^b\nabla_d\theta_{Ac})\theta_b^A dx^d \wedge dx^c - (\nabla^b\nabla_c\theta_{Ab})\theta_d^A dx^d \wedge dx^c - \\ &\quad - (\nabla^b\nabla_b\theta_{Ad})\theta_c^A dx^d \wedge dx^c - (\nabla_d\theta_{Ac})(\nabla^b\theta_b^A) dx^d \wedge dx^c - (\nabla_c\theta_{Ab})(\nabla^b\theta_d^A) dx^d \wedge dx^c, \end{aligned} \quad (5.84)$$

where we omitted the term  $-(\nabla_b\theta_{Ad})(\nabla^b\theta_c^A) dx^d \wedge dx^c$  which being of the form (5.80) is zero.

The last term in (5.79) can be expressed as follows

$$\begin{aligned} * \theta^A * (d * (*\theta_A \wedge \theta^B) \wedge d\theta_B) &= \frac{1}{2}\theta^{Ab}\nabla_i(\theta_{Aa}\theta^{Ba})(\nabla_j\theta_{Bk})\epsilon^{ijk}\epsilon_{bdc} dx^d \wedge dx^c = \\ &= \theta^{Ab}\nabla_b(\theta_{Aa}\theta^{Ba})(\nabla_d\theta_{Bc}) dx^d \wedge dx^c + \theta^{Ab}\nabla_d(\theta_{Aa}\theta^{Ba})(\nabla_c\theta_{Bb}) dx^d \wedge dx^c + \\ &\quad + \theta^{Ab}\nabla_c(\theta_{Aa}\theta^{Ba})(\nabla_b\theta_{Bd}) dx^d \wedge dx^c. \end{aligned}$$

The last term in the second line above vanishes—indeed, it can be expressed as the following sum

$$\theta^{Ab}(\nabla_d\theta_{Aa})\theta^{Ba}(\nabla_c\theta_{Bb}) dx^d \wedge dx^c + (\nabla_d\theta^{Bb})(\nabla_c\theta_{Bb}) dx^d \wedge dx^c$$

and each term in the sum is of the form (5.80). Thus

$$\begin{aligned} * \theta^A * (d * (*\theta_A \wedge \theta^B) \wedge d\theta_B) &= (\nabla_b\theta^{Bb})(\nabla_d\theta_{Bc}) dx^d \wedge dx^c + (\nabla_c\theta^{Bb})(\nabla_b\theta_{Bd}) dx^d \wedge dx^c + \\ &\quad + (\nabla_b\theta_{Aa})(\nabla_d\theta_{Bc})\theta^{Ab}\theta^{Ba} dx^d \wedge dx^c + (\nabla_c\theta_{Aa})(\nabla_b\theta_{Bd})\theta^{Ab}\theta^{Ba} dx^d \wedge dx^c. \end{aligned} \quad (5.85)$$

Now we are ready to gather all the results (5.81)–(5.85) to show that (5.79) is zero. To make the task easier let us note that we obtained three kinds of expressions: (i) ones containing the second derivatives of  $\theta_a^A$ , (ii) ones quadratic in the first derivatives of  $\theta_a^A$  and (iii) ones quadratic in the first derivatives and quadratic in  $\theta_a^A$  (of course, here we take into account only the explicit form of the expressions i.e. we neglect the dependence of  $\nabla_a$  on the components  $\theta_a^A$  and their derivatives).

Expressions containing the second derivatives of  $\theta_a^A$  appear in (5.83) and (5.84) and they read

$$\begin{aligned} & -(\nabla^b \nabla_b \theta_{Bc}) \theta_d^B dx^d \wedge dx^c + (\nabla_d \nabla_a \theta_{Bc}) \theta^{Ba} dx^d \wedge dx^c + (\nabla^b \nabla_c \theta_{Bb}) \theta_d^B dx^d \wedge dx^c - \\ & -(\nabla^b \nabla_d \theta_{Ac}) \theta_b^A dx^d \wedge dx^c - (\nabla^b \nabla_c \theta_{Ab}) \theta_d^A dx^d \wedge dx^c - (\nabla^b \nabla_b \theta_{Ad}) \theta_c^A dx^d \wedge dx^c \end{aligned}$$

We see now that the first and the last term sum up to zero, similarly do the third and the fifth ones. The sum of the remaining second and fourth terms can be expressed as

$$\begin{aligned} [(\nabla_d \nabla_a - \nabla_a \nabla_d) \theta_B^b] \theta^{Ba} q_{bc} dx^d \wedge dx^c &= R^b{}_{eda} \theta_B^e \theta^{Ba} q_{bc} dx^d \wedge dx^c = \\ &= R^b{}_{eda} q^{ea} q_{bc} dx^d \wedge dx^c = R_{ceda} q^{ea} dx^d \wedge dx^c = R_{cd} dx^d \wedge dx^c = 0, \end{aligned}$$

where we used the Riemann tensor  $R^b{}_{eda}$  of the Levi-Civita connection compatible with  $q$  to express the commutator  $(\nabla_d \nabla_a - \nabla_a \nabla_d)$ . Note that in the last step we used the symmetry of the Ricci tensor  $R_{cd}$ .

Terms quadratic in the first derivatives of  $\theta_a^A$  appear in (5.84) and (5.85):

$$\begin{aligned} & -(\nabla_d \theta_{Ac})(\nabla^b \theta_b^A) dx^d \wedge dx^c - (\nabla_c \theta_{Ab})(\nabla^b \theta_d^A) dx^d \wedge dx^c + (\nabla_b \theta^{Bb})(\nabla_d \theta_{Bc}) dx^d \wedge dx^c + \\ & + (\nabla_c \theta^{Bb})(\nabla_b \theta_{Bd}) dx^d \wedge dx^c \end{aligned}$$

It is easy to see that the first and the third terms sum up to zero, similarly do the second and fourth ones.

Let us finally consider the terms quadratic in the first derivatives of  $\theta_a^A$  and quadratic in  $\theta_a^A$ —there are eight of them and they can be grouped into pairs such that each pair sums up to zero. These pairs are:

1. the first term at the r.h.s. of (5.81) and the fourth one at the r.h.s. of (5.83),
2. the second term at the r.h.s. of (5.81) and the third one at the r.h.s. of (5.85) (apply (5.4) to the latter term),
3. the first term at the r.h.s. of (5.82) and the fifth one at the r.h.s. of (5.83),
4. the second term at the r.h.s. of (5.82) and the fourth one at the r.h.s. of (5.85) (apply (5.4) to the latter term).

In this way we demonstrated that (5.79) is zero for every  $M$ ,  $b$  and  $\theta^A$ . Thus the formula (5.78) turns into the final expression of the Poisson bracket of  $R(b)$  and  $S(M)$ :

$$\{R(b), S(M)\} = -R\left(M[\theta^B * (p_B \wedge b) - \frac{1}{2} b * (p_B \wedge \theta^B) + d\xi_A * (b \wedge \theta^A)]\right) - B\left(* (dM \wedge b)\right) \quad (5.86)$$

which coincides with (3.19).

## 5.5 Poisson bracket of $B(a)$ and $S(M)$

The bracket

$$\{B(a), S(M)\} = \sum_{i=1}^2 \sum_{j=1}^3 \{B_i(a), S_j(M)\} \quad (5.87)$$

will be calculated in a similar way to  $\{R(b), S(M)\}$ .

### 5.5.1 Terms quadratic in $p_A$

The only term in (5.87) quadratic in  $p_A$  is

$$\{B_2(a), S_1(M)\} = \int_{\Sigma} \frac{\delta B_2(a)}{\delta \theta^A} \wedge \frac{\delta S_1(M)}{\delta p_A} - \frac{\delta S_1(M)}{\delta \theta^A} \frac{\delta B_2(a)}{\delta p_A}. \quad (5.88)$$

The first term of the r.h.s. of this equation turns out to be zero. To show this let us express the term as follows

$$\begin{aligned} \frac{\delta B_2(a)}{\delta \theta^A} \wedge \frac{\delta S_1(M)}{\delta p_A} &= -\frac{M}{2} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge \theta^E * (a \wedge p_D) * (p_E \wedge \theta^A) + \\ &+ \frac{M}{4} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge \theta^A \wedge * (a \wedge p_D) * (p_E \wedge \theta^E) + [(*\xi^D) \wedge *_A' (a \wedge p_D)] \wedge \frac{\delta S_1(M)}{\delta p_A}. \end{aligned} \quad (5.89)$$

Our strategy now is to restore in each term above the function  $\xi^A$  which originally appears in  $B_2(a)$ . Thus by virtue of (5.51)

$$\begin{aligned} -\frac{M}{2} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge \theta^E * (a \wedge p_D) * (p_E \wedge \theta^A) &= M(*\xi^D)(\vec{\theta}_A \lrcorner \theta^E) * (a \wedge p_D) * (p_E \wedge \theta^A) = \\ &= Ma \wedge \xi^D p_D * (p_A \wedge \theta^A). \end{aligned} \quad (5.90)$$

The second term at the r.h.s of (5.89)

$$\frac{M}{4} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge \theta^A \wedge * (a \wedge p_D) * (p_E \wedge \theta^E) = -\frac{3M}{2} a \wedge \xi^D p_D * (p_A \wedge \theta^A). \quad (5.91)$$

Due to (5.59)

$$[(*\xi^D) \wedge *_A' (a \wedge p_D)] \wedge \frac{\delta S_1(M)}{\delta p_A} = \frac{M}{2} a \wedge \xi^D p_D * (p_A \wedge \theta^A). \quad (5.92)$$

Gathering the three results (5.90), (5.91) and (5.92) we see that, indeed, the first term on the r.h.s. of (5.88) is zero.

The second term at the r.h.s. of (5.88) requires only some simple transformations after which we obtain a simple expression for terms in (5.87) quadratic in the momentum:

$$\{B_2(a), S_1(M)\} = - \int_{\Sigma} M[\theta^B * (p_B \wedge a) - \frac{1}{2} a * (p_B \wedge \theta^B)] \wedge \xi^A p_A. \quad (5.93)$$

### 5.5.2 Terms linear in $p_A$

It turns out that  $\{B_1(a), S_1(M)\}$  and  $\{B_2(a), S_2(M)\}$  give terms linear in  $p_A$ . The first of the two brackets can be calculated as follows

$$\begin{aligned} \{B_1(a), S_1(M)\} = \int_{\Sigma} [-a \wedge *d\theta_A + d*(a \wedge \theta_A)] \wedge M[\theta^B * (p_B \wedge \theta^A) - \frac{1}{2}\theta^A * (p_B \wedge \theta^B)] + \\ + [(a \wedge \theta^B) \wedge *_A' d\theta_B] \wedge \frac{\delta S_1(M)}{\delta p_A} \end{aligned}$$

Using (5.53) after some simple algebra we obtain

$$\begin{aligned} \{B_1(a), S_1(M)\} = \int_{\Sigma} Ma \wedge \theta^B \wedge *d\theta_A * (p_B \wedge \theta^A) + Md[* (a \wedge \theta_A)] \wedge \theta^B * (p_B \wedge \theta^A) - \\ - Ma \wedge \theta^B \wedge *d\theta_B * (p_A \wedge \theta^A) - \frac{M}{2}d[* (a \wedge \theta_A)] \wedge \theta^A * (p_B \wedge \theta^B) + \\ + M * (a \wedge \theta^B \wedge \theta^A) * p_A \wedge d\theta_B - Ma \wedge *p_A \wedge \theta^B * (d\theta_B \wedge \theta^A). \end{aligned}$$

The next bracket reads

$$\begin{aligned} \{B_2(a), S_2(M)\} = \int_{\Sigma} -\frac{1}{2}\epsilon^D{}_{BCA}\theta^B \wedge \theta^C \wedge (\xi^A dM + Md\xi^A) * (a \wedge p_D) + \\ + M[(\xi^B) \wedge *_A' (a \wedge p_B)] \wedge d\xi^A - \frac{M}{2}(*dp_D)\epsilon^D{}_{BCA}\theta^B \wedge \theta^C \wedge a\xi^A. \quad (5.94) \end{aligned}$$

To simplify the resulting expression let us first consider the two terms above containing  $d\xi^A$ —the first of the terms can be transformed by means of (5.51):

$$-\frac{1}{2}\epsilon^D{}_{BCA}\theta^B \wedge \theta^C \wedge (Md\xi^A * (a \wedge p_D)) = M(*\xi^D)\vec{\theta}_A \lrcorner d\xi^A * (a \wedge p_D) = M(\vec{\theta}_A \lrcorner d\xi^A)a \wedge \xi^D p_D,$$

On the other hand by virtue of (5.58) the other term

$$M[(\xi^B) \wedge *_A' (a \wedge p_B)] \wedge d\xi^A = -M(*\xi^B) \wedge *(a \wedge p_B)(\vec{\theta}_A \lrcorner d\xi^A) = -M(\vec{\theta}_A \lrcorner d\xi^A)a \wedge \xi^B p_B.$$

Thus the sum of the two terms in (5.94) containing  $d\xi^A$  is zero. Consequently,

$$\begin{aligned} \{B_2(a), S_2(M)\} = \int_{\Sigma} -\frac{1}{2}\epsilon^D{}_{BCA}\theta^B \wedge \theta^C \xi^A \wedge (dM * (a \wedge p_D) + Ma * dp_D) = \\ = \int_{\Sigma} *\theta^D \wedge (dM * (a \wedge p_D) + Ma * dp_D), \quad (5.95) \end{aligned}$$

where we applied (5.15).

Finally, the terms in (5.87) linear in  $p_A$  read

$$\begin{aligned}
\{B_1(a), S_1(M)\} + \{B_2(a), S_2(M)\} = & \int_{\Sigma} * \theta^D \wedge (dM * (a \wedge p_D) + Ma * dp_D) + \\
& + Ma \wedge \theta^B \wedge * d\theta_A * (p_B \wedge \theta^A) + Md[* (a \wedge \theta_A)] \wedge \theta^B * (p_B \wedge \theta^A) - \\
& - Ma \wedge \theta^B \wedge * d\theta_B * (p_A \wedge \theta^A) - \frac{M}{2} d[* (a \wedge \theta_A)] \wedge \theta^A * (p_B \wedge \theta^B) + \\
& + M * (a \wedge \theta^B \wedge \theta^A) * p_A \wedge d\theta_B - Ma \wedge * p_A \wedge \theta^B * (d\theta_B \wedge \theta^A). \quad (5.96)
\end{aligned}$$

### 5.5.3 Terms independent of $p_A$

The remaining three brackets  $\{B_1(a), S_3(M)\}$ ,  $\{B_1(a), S_2(M)\}$  and  $\{B_2(a), S_3(M)\}$  give terms independent of  $p_A$ . The first bracket is zero since both  $B_1(a)$  and  $S_3(M)$  do not depend on  $p_A$ . The second bracket

$$\begin{aligned}
\{B_1(a), S_2(M)\} = & \int_{\Sigma} [-a \wedge * d\theta_A + (a \wedge \theta^B) \wedge *'_A d\theta_B] \wedge (dM \xi^A + M d\xi^A) = \\
= & \int_{\Sigma} -dM \wedge a \wedge * (\xi^A d\theta_A) - Ma \wedge * d\theta_A \wedge d\xi^A + M(a \wedge \theta^B) * (d\theta_B \wedge \theta_A) \wedge d\xi^A - \\
& - M * (* d\theta_B \wedge \theta_A) \wedge * (a \wedge \theta^B) \wedge d\xi^A,
\end{aligned}$$

where in the second step we used the second line of (5.52) and (4.9).

The third bracket

$$\begin{aligned}
\{B_2(a), S_3(M)\} = & - \int_{\Sigma} d \left( M \theta^B * (d\theta_B \wedge \theta_A) - \frac{M}{2} \theta_A * (d\theta_B \wedge \theta^B) \right) \wedge a \xi^A + \\
+ & M \left( d\theta_B * (d\theta_A \wedge \theta^B) - \frac{1}{2} d\theta_A * (d\theta_B \wedge \theta^B) \right) \wedge a \xi^A = - \int_{\Sigma} -Ma \wedge \theta^B \wedge \xi^A d * (d\theta_B \wedge \theta_A) + \\
& + Ma \wedge d\theta_B \xi^A * (d\theta_A \wedge \theta^B) - Ma \wedge \xi^A d\theta_A * (d\theta_B \wedge \theta^B).
\end{aligned}$$

Thus the terms in (5.87) independent of  $p_A$  read

$$\begin{aligned}
\{B_1(a), S_2(M)\} + \{B_2(a), S_3(M)\} = & \int_{\Sigma} -dM \wedge a \wedge * (\xi^A d\theta_A) - Ma \wedge * d\theta_A \wedge d\xi^A + \\
& + Ma \wedge \theta^B * (d\theta_B \wedge \theta_A) \wedge d\xi^A - M * (* d\theta_B \wedge \theta_A) \wedge * (a \wedge \theta^B) \wedge d\xi^A + \\
+ & Ma \wedge \theta^B \wedge \xi^A d * (d\theta_B \wedge \theta_A) - Ma \wedge d\theta_B \xi^A * (d\theta_A \wedge \theta^B) + Ma \wedge \xi^A d\theta_A * (d\theta_B \wedge \theta^B). \quad (5.97)
\end{aligned}$$

### 5.5.4 Isolating constraints

Again, our goal now is to express the bracket

$$\{B(a), S(M)\} = \text{the terms (5.93) quadratic in } p_A + \\ + \text{the terms (5.96) linear in } p_A + \text{the terms (5.97) independent of } p_A$$

as a combination of the constraints.

**Terms quadratic in  $p_A$**  We immediately see that the formula (5.93) can be expressed as

$$-B\left(M[\theta^B * (p_B \wedge a) - \frac{1}{2}a * (p_B \wedge \theta^B)]\right) + \int_{\Sigma} M[\theta^B * (p_B \wedge a) - \frac{1}{2}a * (p_B \wedge \theta^B)] \wedge \theta^A \wedge *d\theta_A, \quad (5.98)$$

hence

$$\{B(a), S(M)\} = -B\left(M[\theta^B * (p_B \wedge a) - \frac{1}{2}a * (p_B \wedge \theta^B)]\right) + \\ + \text{terms linear in } p_A + \text{the terms (5.97) independent of } p_A, \quad (5.99)$$

where now “terms linear in  $p_A$ ” mean the last term in (5.98) and (5.96).

**Terms linear in  $p_A$**  According to the last statement of the previous paragraph the remaining terms linear in  $p_A$  read

$$\int_{\Sigma} M[\theta^B * (p_B \wedge a) - \frac{1}{2}a * (p_B \wedge \theta^B)] \wedge \theta^A \wedge *d\theta_A + *\theta^D \wedge (dM * (a \wedge p_D) + Ma * dp_D) + \\ + Ma \wedge \theta^B \wedge *d\theta_A * (p_B \wedge \theta^A) + Md[* (a \wedge \theta_A)] \wedge \theta^B * (p_B \wedge \theta^A) - \\ - Ma \wedge \theta^B \wedge *d\theta_B * (p_A \wedge \theta^A) - \frac{M}{2}d[* (a \wedge \theta_A)] \wedge \theta^A * (p_B \wedge \theta^B) + \\ + M * (a \wedge \theta^B \wedge \theta^A) * p_A \wedge d\theta_B - Ma \wedge *p_A \wedge \theta^B * (d\theta_B \wedge \theta^A). \quad (5.100)$$

Let us now transform the terms containing  $dM$  and  $dp_D$  appearing in the first line of (5.100):

$$\int_{\Sigma} *\theta^D \wedge dM * (a \wedge p_D) = \int_{\Sigma} *(*dM \wedge \theta^D)a \wedge p_D \int_{\Sigma} = (\vec{\theta}^D \lrcorner dM)a \wedge p_D = \\ = \int_{\Sigma} dM(\vec{\theta}^D \lrcorner a) \wedge p_D - dM \wedge a \vec{\theta}^D \lrcorner p_D = \int_{\Sigma} dM(\vec{\theta}^D \lrcorner a) \wedge p_D + *(dM \wedge a)\theta^A \wedge *p_A = \\ = R\left(* (dM \wedge a)\right) + \int_{\Sigma} dM(\vec{\theta}^D \lrcorner a) \wedge p_D + *(dM \wedge a) \wedge \xi^A d\theta_A,$$

where we used (4.9). On the other hand the term with  $dp_D$

$$\begin{aligned}\int_{\Sigma} * \theta^D \wedge Ma * dp_D &= \int_{\Sigma} M * (*a \wedge \theta^D) dp_D = - \int_{\Sigma} d[M * (*a \wedge \theta^D)] \wedge p_D = \\ &= - \int_{\Sigma} dM(\vec{\theta}^D \lrcorner a) \wedge p_D + Md * (*a \wedge \theta^D) \wedge p_D.\end{aligned}$$

Consequently, the sum of the two terms

$$\begin{aligned}\int_{\Sigma} * \theta^D \wedge (dM * (a \wedge p_D) + Ma * dp_D) &= R\left(* (dM \wedge a)\right) + \int_{\Sigma} -Md * (*a \wedge \theta^A) \wedge p_A + \\ &\quad + dM \wedge a \wedge *(\xi^A d\theta_A). \quad (5.101)\end{aligned}$$

The result just obtained means that (5.99) can be reexpressed as

$$\begin{aligned}\{B(a), S(M)\} &= -B\left(M[\theta^B * (p_B \wedge a) - \frac{1}{2}a * (p_B \wedge \theta^B)]\right) + R\left(* (dM \wedge a)\right) \\ &\quad + \text{terms linear in } p_A + \text{terms independent of } p_A, \quad (5.102)\end{aligned}$$

where now (i) “terms linear in  $p_A$ ” mean the second term at the r.h.s. of (5.101) and (5.100) except the terms containing  $dM$  and  $dp_D$  and (ii) “terms independent of  $p_A$ ” mean the last term in (5.101) and the terms (5.97).

Note now that the form of constraints at the r.h.s. of (5.102) we managed to isolate so far resemble closely the form of the constraints at the r.h.s. of (5.86). Thus we *postulate* that

$$\{B(a), S(M)\} = -B\left(M[\theta^B * (p_B \wedge a) - \frac{1}{2}a * (p_B \wedge \theta^B) + d\xi_B * (a \wedge * \theta^B)]\right) + R\left(* (dM \wedge a)\right). \quad (5.103)$$

To justify the postulate we will proceed as follows: we will add to the r.h.s. of (5.102) zero expressed as

$$\begin{aligned}0 &= -B\left(Md\xi_B * (a \wedge * \theta^B)\right) + B\left(Md\xi_B * (a \wedge * \theta^B)\right) = -B\left(Md\xi_B * (a \wedge * \theta^B)\right) + \\ &\quad + \int_{\Sigma} Md\xi_B * (a \wedge * \theta^B)(\theta^A \wedge * d\theta_A + \xi^A p_A) = -B\left(Md\xi_B * (a \wedge * \theta^B)\right) + \\ &\quad + \int_{\Sigma} Md\xi_B * (a \wedge * \theta^B) \theta^A \wedge * d\theta_A + Md * (* \theta_B \wedge \theta^A) \wedge p_A * (a \wedge * \theta^B) \quad (5.104)\end{aligned}$$

(here in the last step we used (5.55)). Next we will show that all the remaining terms linear in  $p_A$  sum up to zero, and that similarly do all the remaining terms independent of the momentum (note that now the description of the term linear in and independent

of the momentum given just below Equation (5.102) has to be completed by taking into account the two last term in (5.104)).

To demonstrate that all the remaining terms linear in  $p_A$

$$\begin{aligned}
& \int_{\Sigma} M \theta^B * (p_B \wedge a) \wedge \theta^A \wedge * d\theta_A - \frac{M}{2} a * (p_B \wedge \theta^B) \wedge \theta^A \wedge * d\theta_A - M d * (*a \wedge \theta^A) \wedge p_A + \\
& + M a \wedge \theta^B \wedge * d\theta_A * (p_B \wedge \theta^A) + M d[* (a \wedge \theta_A)] \wedge \theta^B * (p_B \wedge \theta^A) - \\
& - M a \wedge \theta^B \wedge * d\theta_B * (p_A \wedge \theta^A) - \frac{M}{2} d[* (a \wedge \theta_A)] \wedge \theta^A * (p_B \wedge \theta^B) + \\
& + M * (a \wedge \theta^B \wedge \theta^A) * p_A \wedge d\theta_B - M a \wedge * p_A \wedge \theta^B * (d\theta_B \wedge \theta^A) + M d * (*\theta_B \wedge \theta^A) \wedge p_A * (a \wedge * \theta^B).
\end{aligned} \tag{5.105}$$

sum up to zero let us first perform some transformations. First we are going to show that the first, fourth, sixth and eighth terms above give together zero. To this end let us transform the eighth one as follows

$$\begin{aligned}
M * (a \wedge \theta^B \wedge \theta^A) * p_A \wedge d\theta_B &= -M[\vec{\theta}^B \lrcorner * (a \wedge \theta^A)] * p_A \wedge d\theta_B = \\
&= -M * (a \wedge \theta^A) * (p_A \wedge \theta^B) \wedge d\theta_B + M * (a \wedge \theta^A) * p_A \wedge (*d\theta_B \wedge \theta^B),
\end{aligned}$$

where in the second step we shifted the contraction  $\vec{\theta}^B \lrcorner$  and applied (4.9). Let us now transform the last term above in an analogous way:

$$\begin{aligned}
M * (a \wedge \theta^A) * p_A \wedge (*d\theta_B \wedge \theta^B) &= M(\vec{\theta}^A \lrcorner * a) \wedge * p_A \wedge (*d\theta_B \wedge \theta^B) = \\
&= -M * a \wedge * (p_A \wedge \theta^A) * (*d\theta_B \wedge \theta^B) + M * a \wedge * p_A * (*d\theta_B \wedge \theta^B \wedge \theta^A) = \\
&= M a \wedge \theta^B \wedge * d\theta_B * (p_A \wedge \theta^A) - M \theta^A * (p_A \wedge a) \wedge \theta^B \wedge * d\theta_B.
\end{aligned}$$

Thus the eighth term

$$\begin{aligned}
M * (a \wedge \theta^B \wedge \theta^A) * p_A \wedge d\theta_B &= -M a \wedge \theta^A \wedge * d\theta_B * (p_A \wedge \theta^B) - \\
&+ M a \wedge \theta^B \wedge * d\theta_B * (p_A \wedge \theta^A) - M \theta^A * (p_A \wedge a) \wedge \theta^B \wedge * d\theta_B
\end{aligned}$$

and indeed the first, fourth, sixth and eighth terms disappear from (5.105).

Let us consider now the second and the seventh terms in (5.105). After a slight transformation of the second one their sum can be expressed as

$$\begin{aligned}
& -\frac{M}{2} [* (a \wedge \theta^A) \wedge d\theta_A + d * (a \wedge \theta^A) \wedge \theta_A] * (p_B \wedge \theta^B) = \\
& = -\frac{M}{2} [2 * (a \wedge \theta^A) \wedge d\theta_A - d(\theta_A * (a \wedge \theta^A))] * (p_B \wedge \theta^B) = \\
& \quad - M [* (a \wedge \theta^A) \wedge d\theta_A - d * a] * (p_B \wedge \theta^B),
\end{aligned}$$



where in the last step we used (5.13).

Now the terms (5.105) can be reexpressed in a simpler form as

$$\begin{aligned} & \int_{\Sigma} -M * (a \wedge \theta^A) \wedge d\theta_A * (p_B \wedge \theta^B) + Md * a * (p_B \wedge \theta^B) - Md * (*a \wedge \theta^A) \wedge p_A + \\ & + Md[* (a \wedge \theta_A)] \wedge \theta^B * (p_B \wedge \theta^A) - Ma \wedge * p_A \wedge \theta^B * (d\theta_B \wedge \theta^A) + Md * (*\theta_B \wedge \theta^A) \wedge p_A * (a \wedge *\theta^B). \end{aligned}$$

Note that in each term above one can isolate the factor  $Mp_B$  obtaining thereby

$$\begin{aligned} & \int_{\Sigma} Mp_B \wedge \left( -\theta^B * [* (a \wedge \theta^A) \wedge d\theta_A] + \theta^B * d * a - d * (*a \wedge \theta^B) + \right. \\ & \left. + \theta^A * [d * (a \wedge \theta_A) \wedge \theta^B] + *(a \wedge \theta^A) * (d\theta_A \wedge \theta^B) + d * (*\theta_A \wedge \theta^B) * (a \wedge *\theta^A) \right). \end{aligned} \quad (5.106)$$

Now we will show by a direct calculation that the terms in the big parenthesis above sum up to zero for every  $\theta^A$  and  $a$ . The first term in (5.106) can be expressed as

$$\begin{aligned} -\theta^B * [* (a \wedge \theta^A) \wedge d\theta_A] &= -\theta_c^B a^a \theta^{Ab} (\nabla_d \theta_{Ae}) \epsilon_{abf} \epsilon^{fde} dx^c = \\ &= -\theta_c^B a^a \theta^{Ab} (\nabla_a \theta_{Ab} - \nabla_b \theta_{Aa}) dx^c = a^a \theta^{Ab} (\nabla_b \theta_{Aa}) \theta_c^B dx^c, \end{aligned} \quad (5.107)$$

where in the second step we used (5.9), and in the last one (5.5).

The second term in (5.106)

$$\theta^B * d * a = (\nabla^a a_a) \theta_c^B dx^c \quad (5.108)$$

by virtue of (5.8).

The third one

$$-d * (*a \wedge \theta^B) = -\nabla_c (\theta^{Ba} a_a) dx^c = -(\nabla_c \theta^{Ba}) a_a dx^c - \theta^{Ba} (\nabla_c a_a) dx^c. \quad (5.109)$$

The fourth term in (5.106) by means of (5.9) can be expressed as

$$\begin{aligned} \theta^A * [d * (a \wedge \theta_A) \wedge \theta^B] &= \nabla_d (a^a \theta_A^b) \theta_e^B \theta_c^A \epsilon_{abf} \epsilon^{dfe} dx^c = -(\nabla_a a^a) \theta_c^B dx^c - \\ &- a^a (\nabla_a \theta_A^b) \theta_b^B \theta_c^A dx^c + (\nabla_c a^a) \theta_a^B dx^c + a^a (\nabla_b \theta_A^b) \theta_a^B \theta_c^A dx^c. \end{aligned} \quad (5.110)$$

The fifth one

$$*(a \wedge \theta^A) * (d\theta_A \wedge \theta^B) = a^a \theta^{Ab} (\nabla_a \theta_{Ae}) \theta_f^B \epsilon_{abc} \epsilon^{def} dx^c.$$

Using (5.9) we obtain six terms and two of them vanish by virtue of (5.5). Thus

$$\begin{aligned} *(a \wedge \theta^A) * (d\theta_A \wedge \theta^B) &= a^a \theta^{Ab} (\nabla_b \theta_{Ac}) \theta_a^B dx^c + a^a \theta^{Ab} (\nabla_c \theta_{Aa}) \theta_b^B dx^c - \\ &- a^a \theta^{Ab} (\nabla_b \theta_{Aa}) \theta_c^B dx^c - a^a \theta^{Ab} (\nabla_a \theta_{Ac}) \theta_b^B dx^c. \end{aligned} \quad (5.111)$$

The last term in (5.106)

$$d * (*\theta_A \wedge \theta^B) * (a \wedge *\theta^A) = \nabla_c (\theta_A^b \theta_b^B) a_a \theta^{Aa} dx^c = (\nabla_c \theta_A^b) \theta_b^B a_a \theta^{Aa} dx^c + (\nabla_c \theta^{Ba}) a_a dx^c. \quad (5.112)$$

Collecting all the results (5.107)–(5.112) we note that we obtain two kinds of terms: ones containing a derivative of  $a^a$  and ones containing a derivative of  $\theta_a^A$ . The terms containing  $\nabla_b a^a$  appear in (5.108), (5.109) and (5.110) and sum up to zero:

$$(\nabla^a a_a) \theta_c^B dx^c - \theta^{Ba} (\nabla_c a_a) dx^c - (\nabla_a a^a) \theta_c^B dx^c + (\nabla_c a^a) \theta_a^B dx^c = 0.$$

Regarding the terms containing  $\nabla_a \theta_b^A$  there are ten of them and they can be grouped into pairs such that each pair sums up to zero. These pairs are:

1. the term at the r.h.s. of (5.107) and the third term at the r.h.s. of (5.111),
2. the first term at the r.h.s. of (5.109) and the last term at the r.h.s. of (5.112),
3. the second term at the r.h.s. of (5.110) and the last term at the r.h.s. of (5.111) (shift the derivative by means of (5.4) in the latter term),
4. the last term at the r.h.s. of (5.110) and the first term at the r.h.s. of (5.111) (shift the derivative by means of (5.4) in the latter term),
5. the second term at the r.h.s. of (5.111) and the first term at the r.h.s. of (5.112) (again shift the derivative by means of (5.4) in the latter term).

**Terms independent of  $p_A$**  Our goal now is to show that all the remaining terms independent of the momentum i.e. the terms (5.97), the last term in (5.101) and the second term at the r.h.s. of (5.104) sum up to zero. Note that the first term in (5.97) cancels the last term in (5.101). Now in all remaining terms there is the factor  $Ma$  and therefore they can be expressed as

$$\begin{aligned} \int_{\Sigma} Ma \wedge \big( & - *d\theta_A \wedge d\xi^A + \theta^B * (d\theta_B \wedge \theta_A) \wedge d\xi^A + \theta^B \wedge *[* (d\theta_B \wedge \theta_A) \wedge d\xi^A] + \\ & + \theta^B \wedge \xi^A d * (d\theta_B \wedge \theta_A) - d\theta_B * (\theta_A \wedge d\xi^A \wedge \theta^B) + \theta_A \wedge d\xi^A * (d\theta_B \wedge \theta^B) + \\ & + *\theta^A * (d\xi_A \wedge \theta^B \wedge *d\theta_B) \big) \end{aligned}$$

By a direct calculation we will demonstrate that the terms in the big parenthesis sum up to zero for all  $\theta^A$ . To be more precise we will show that

$$\begin{aligned} * \Big( & - * d\theta_A \wedge d\xi^A + \theta^B \wedge d\xi^A * (d\theta_B \wedge \theta_A) + \theta^B \wedge *[*(*d\theta_B \wedge \theta_A) \wedge d\xi^A] + \\ & + \theta^B \wedge \xi^A d * (d\theta_B \wedge \theta_A) - d\theta_B * (\theta_A \wedge d\xi^A \wedge \theta^B) + \theta_A \wedge d\xi^A * (d\theta_B \wedge \theta^B) + \\ & + * \theta^A * (d\xi_A \wedge \theta^B \wedge * d\theta_B) \Big) \end{aligned} \quad (5.113)$$

is equal to zero.

The first term in the expression above reads

$$\begin{aligned} - *(*d\theta_A \wedge d\xi^A) &= -(\nabla_a \theta_{Ab})(\nabla^d \xi^A) \epsilon^{abc} \epsilon_{cde} dx^e = -(\nabla_a \theta_{Ab})(\nabla^a \xi^A) dx^b + \\ &+ (\nabla_a \theta_{Ab})(\nabla^b \xi^A) dx^a, \end{aligned} \quad (5.114)$$

where in the last step we used (5.9).

The second term in (5.113)

$$*(\theta^B \wedge d\xi^A) * (d\theta_B \wedge \theta_A) = \theta^{Ba} (\nabla^b \xi^A) (\nabla_c \theta_{Bd}) \theta_{Ae} \epsilon^{cde} \epsilon_{abf} dx^f.$$

By virtue of (5.9) it can be expressed as a sum of six terms. Two of them vanish due to (5.5) and we obtain

$$\begin{aligned} *(\theta^B \wedge d\xi^A) * (d\theta_B \wedge \theta_A) &= \theta^{Ba} (\nabla^b \xi^A) (\nabla_a \theta_{Bb}) \theta_{Af} dx^f + \theta^{Ba} (\nabla^b \xi^A) (\nabla_b \theta_{Bf}) \theta_{Aa} dx^f - \\ &- \theta^{Ba} (\nabla^b \xi^A) (\nabla_a \theta_{Bf}) \theta_{Ab} dx^f - \theta^{Ba} (\nabla^b \xi^A) (\nabla_f \theta_{Bb}) \theta_{Aa} dx^f. \end{aligned} \quad (5.115)$$

The third term reads

$$*(\theta^B \wedge *[*(*d\theta_B \wedge \theta_A) \wedge d\xi^A]) = \theta_A^a (\nabla_a \theta_{Bb} - \nabla_b \theta_{Ba}) (\nabla_c \xi^A) \theta^{Be} \epsilon^{bcd} \epsilon_{edf} dx^f.$$

After applying (5.9) we see that one term vanishes by virtue of (5.5) and we get

$$\begin{aligned} *(\theta^B \wedge *[*(*d\theta_B \wedge \theta_A) \wedge d\xi^A]) &= \theta_A^a (\nabla_b \theta_{Ba}) (\nabla_c \xi^A) \theta^{Bb} dx^c + \\ &+ \theta_A^a (\nabla_a \theta_{Bb}) (\nabla_c \xi^A) \theta^{Bc} dx^b - \theta_A^a (\nabla_b \theta_{Ba}) (\nabla_c \xi^A) \theta^{Bc} dx^b. \end{aligned} \quad (5.116)$$

The fourth term in (5.113)

$$*[\theta^B \wedge \xi^A d * (d\theta_B \wedge \theta_A)] = \theta^{Ba} \xi^A (\nabla^b \theta_{Ad}) (\nabla_e \theta_{Bf}) \epsilon^{def} \epsilon_{abc} dx^c.$$

Again using (5.9) we get six terms, two of them vanish due to (5.5) and we obtain thereby

$$\begin{aligned} *[\theta^B \wedge \xi^A d * (d\theta_B \wedge \theta_A)] &= \theta^{Ba} \xi^A (\nabla^b \theta_{Aa}) (\nabla_b \theta_{Bc}) dx^c + \theta^{Ba} \xi^A (\nabla^b \theta_{Ac}) (\nabla_a \theta_{Bb}) dx^c - \\ &- \theta^{Ba} \xi^A (\nabla^b \theta_{Ab}) (\nabla_a \theta_{Bc}) dx^c - \theta^{Ba} \xi^A (\nabla^b \theta_{Aa}) (\nabla_c \theta_{Bb}) dx^c. \end{aligned} \quad (5.117)$$

The fifth term

$$- * d\theta_B * (\theta_A \wedge d\xi^A \wedge \theta^B) = -\theta_{Aa}(\nabla_b \xi^A) \theta_c^B (\nabla^d \theta_B^e) \epsilon^{abc} \epsilon_{def} dx^f.$$

Applying (5.9) we again obtain six terms, two of them vanish by virtue of (5.5) and we are left with the following expression

$$\begin{aligned} - * d\theta_B * (\theta_A \wedge d\xi^A \wedge \theta^B) = & -\theta_{Aa}(\nabla_b \xi^A) \theta_c^B (\nabla^a \theta_B^b) dx^c - \theta_{Aa}(\nabla_b \xi^A) \theta_c^B (\nabla^c \theta_B^a) dx^b + \\ & + \theta_{Aa}(\nabla_b \xi^A) \theta_c^B (\nabla^b \theta_B^a) dx^c + \theta_{Aa}(\nabla_b \xi^A) \theta_c^B (\nabla^c \theta_B^b) dx^a. \end{aligned} \quad (5.118)$$

The sixth term in (5.113) reads

$$\begin{aligned} * (\theta_A \wedge d\xi^A) * (d\theta_B \wedge \theta^B) = & \theta_A^a (\nabla^b \xi^A) (\nabla_d \theta_{Be}) \theta_f^B \epsilon^{def} \epsilon_{abc} dx^c = \theta_A^a (\nabla^b \xi^A) (\nabla_a \theta_{Bb}) \theta_c^B dx^c + \\ & + \theta_A^a (\nabla^b \xi^A) (\nabla_b \theta_{Bc}) \theta_a^B dx^c + \theta_A^a (\nabla^b \xi^A) (\nabla_c \theta_{Ba}) \theta_b^B dx^c - \theta_A^a (\nabla^b \xi^A) (\nabla_b \theta_{Ba}) \theta_c^B dx^c - \\ & - \theta_A^a (\nabla^b \xi^A) (\nabla_a \theta_{Bc}) \theta_b^B dx^c - \theta_A^a (\nabla^b \xi^A) (\nabla_c \theta_{Bb}) \theta_a^B dx^c. \end{aligned} \quad (5.119)$$

Finally, the last term in (5.113)

$$\theta^A * (d\xi_A \wedge \theta^B \wedge * d\theta_B) = (\nabla_d \xi_A) \theta_e^B (\nabla^a \theta_B^b) \theta_c^A \epsilon_{abf} \epsilon^{def} dx^c = -(\nabla_b \xi_A) \theta_a^B (\nabla^a \theta_B^b) \theta_c^A dx^c, \quad (5.120)$$

where in the last step we used (5.9) and (5.5).

In this way we managed to express (5.113) in terms of the components  $\theta_a^A$  and  $\xi^A$  and their covariant derivatives obtaining altogether twenty four terms (5.114)–(5.120). As before those terms can be grouped into pairs such that terms in each pair sum up to zero. Let us now enumerate the pairs:

1. the first term at the r.h.s. of (5.114) and the second term at the r.h.s. of (5.115). Here the latter term needs the following transformation:

$$\begin{aligned} \theta^{Ba} (\nabla^b \xi^A) (\nabla_b \theta_{Bf}) \theta_{Aa} dx^f &= (\nabla^b \xi^A) (\nabla_b \theta_{Bf}) (\delta^B_A + \xi^B \xi_A) dx^f = \\ &= (\nabla^b \xi^A) (\nabla_b \theta_{Bf}) dx^f, \end{aligned}$$

where in the first step we used (4.23), the second one holds true by virtue of  $(\nabla^b \xi^A) \xi_A = 0$ .

2. the second term at the r.h.s. of (5.114) and the fourth term at the r.h.s. of (5.115) (the latter term needs a transformation analogous to that shown above),
3. the first term at the r.h.s. of (5.115) and the second term at the r.h.s. of (5.117)—in the latter term one should shift the derivative to  $\xi^A$  according to the identity  $\xi^A (\nabla_a \theta_{Ab}) = -(\nabla_a \xi^A) \theta_{AB}$ ,

4. the third term at the r.h.s. of (5.115) and the third term at the r.h.s. of (5.117) (shift the derivative to  $\xi^A$  in the latter term),
5. the first term at the r.h.s. of (5.116) and the second term at the r.h.s. of (5.118),
6. the second term at the r.h.s. of (5.116) and the first term at the r.h.s. of (5.119) (apply (5.4) to the latter term),
7. the third term at the r.h.s. of (5.116) and the third term at the r.h.s. of (5.119),
8. the first term at the r.h.s. of (5.117) and the second term at the r.h.s. of (5.119) (shift the derivative to  $\xi^A$  in the former term),
9. the fourth term at the r.h.s. of (5.117) and the sixth term at the r.h.s. of (5.119) (shift the derivative to  $\xi^A$  in the former term),
10. the first term at the r.h.s. of (5.118) and the fifth term at the r.h.s. of (5.119) (apply (5.4) to the latter term),
11. the third term at the r.h.s. of (5.118) and the fourth term at the r.h.s. of (5.119),
12. the fourth term at the r.h.s. of (5.118) and the term at the r.h.s. of (5.120).

Thus we managed to demonstrate that all the remaining terms (5.113) independent of  $p_A$  sum up to zero and thereby proved the postulate (5.103) which coincides with (3.18).

## 5.6 Poisson brackets of $V(\vec{M})$

In this subsection we will derive the brackets (3.20).

The functional derivatives of the smeared scalar constraint  $V(\vec{M})$  (see (3.13)) are of the following form [17]:

$$\frac{\delta V(\vec{M})}{\delta p_A} = \mathcal{L}_{\vec{M}} \theta^A, \quad \frac{\delta V(\vec{M})}{\delta \theta^A} = -\mathcal{L}_{\vec{M}} p_A. \quad (5.121)$$

It was shown in [17] that

$$\{V(\vec{M}), V(\vec{M}')\} = V([\vec{M}, \vec{M}']),$$

where  $[\vec{M}, \vec{M}']$  denotes the Lie bracket of the vector fields  $\vec{M}, \vec{M}'$  on  $\Sigma$ .

Derivations of brackets of  $V(\vec{M})$  and the other constraints will be based on the following formula [17]:

$$\mathcal{L}_{\vec{M}}(\alpha \wedge * \beta) = \mathcal{L}_{\vec{M}} \alpha \wedge * \beta + \alpha \wedge * \mathcal{L}_{\vec{M}} \beta + \mathcal{L}_{\vec{M}} \theta^A \wedge (\alpha \wedge *_A' \beta). \quad (5.122)$$

We will also apply the following well known property of the Lie derivative  $\mathcal{L}_{\vec{M}}\alpha$  of any three-form  $\alpha$ :

$$\int_{\Sigma} \mathcal{L}_{\vec{M}}\alpha = 0. \quad (5.123)$$

### 5.6.1 Poisson bracket of $V(\vec{M})$ and $S(M)$

To calculate the bracket  $\{V(\vec{M}), S(M)\}$  we will use the split of  $S(M)$  into the three functionals (5.24):

$$\{V(\vec{M}), S(M)\} = \sum_{i=1}^3 \{V(\vec{M}), S_i(M)\}.$$

To calculate the first term  $\{V(\vec{M}), S_1(M)\}$  at the r.h.s. of this equation let us split  $S_1(M)$  into

$$S_{11}(M) := \int_{\Sigma} \frac{M}{2} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) \quad \text{and} \quad S_{12}(M) := - \int_{\Sigma} \frac{M}{4} (p_A \wedge \theta^A) \wedge * (p_B \wedge \theta^B)$$

and consider the bracket  $\{V(\vec{M}), S_{11}(M)\}$ . The functional derivatives of  $S_{11}(M)$  read

$$\begin{aligned} \frac{\delta S_{11}(M)}{\delta \theta^A} &= M \left( p_B * (p_A \wedge \theta^B) + \frac{1}{2} (p_C \wedge \theta^B) \wedge *_A' (p_B \wedge \theta^C) \right), \\ \frac{\delta S_{11}(M)}{\delta p_A} &= M \theta^B * (p_B \wedge \theta^A). \end{aligned}$$

Consequently,

$$\begin{aligned} \{V(\vec{M}), S_{11}(M)\} &= \int_{\Sigma} -\mathcal{L}_{\vec{M}} p_A \wedge M \theta^B * (p_B \wedge \theta^A) - \\ &\quad - \mathcal{L}_{\vec{M}} \theta^A \wedge M \left( p_B * (p_A \wedge \theta^B) + \frac{1}{2} (p_C \wedge \theta^B) \wedge *_A' (p_B \wedge \theta^C) \right) = \\ &= - \int_{\Sigma} M \left( \mathcal{L}_{\vec{M}} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) + \mathcal{L}_{\vec{M}} \theta^A \wedge \frac{1}{2} [(p_C \wedge \theta^B) \wedge *_A' (p_B \wedge \theta^C)] \right). \end{aligned}$$

It is easy to see that the first term in the last line above

$$M \mathcal{L}_{\vec{M}} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) = \frac{M}{2} \left( \mathcal{L}_{\vec{M}} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) + (p_A \wedge \theta^B) \wedge * \mathcal{L}_{\vec{M}} (p_B \wedge \theta^A) \right).$$

This fact together with (5.122) allow us to write

$$\begin{aligned} \{V(\vec{M}), S_{11}(M)\} &= - \int_{\Sigma} M \mathcal{L}_{\vec{M}} \left( \frac{1}{2} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) \right) = \\ &= \int_{\Sigma} (\mathcal{L}_{\vec{M}} M) \left( \frac{1}{2} (p_A \wedge \theta^B) \wedge * (p_B \wedge \theta^A) \right) = S_{11}(\mathcal{L}_{\vec{M}} M), \end{aligned}$$

where the second equality holds by virtue of (5.123). Similarly,  $\{V(\vec{M}), S_{12}(M)\} = S_{12}(\mathcal{L}_{\vec{M}}M)$ . Thus

$$\{V(\vec{M}), S_1(M)\} = S_1(\mathcal{L}_{\vec{M}}M).$$

In analogous way one can show that  $\{V(\vec{M}), S_i(M)\} = S_i(\mathcal{L}_{\vec{M}}M)$  for  $i = 2, 3$  and consequently

$$\{V(\vec{M}), S(M)\} = S(\mathcal{L}_{\vec{M}}M).$$

### 5.6.2 Poisson bracket of $V(\vec{M})$ and the primary constraints

Here we explicitly calculate the bracket  $\{V(\vec{M}), B(a)\}$ . The bracket  $\{V(\vec{M}), R(b)\}$  can be calculated similarly.

Obviously,

$$\{V(\vec{M}), B(a)\} = \{V(\vec{M}), B_1(a)\} + \{V(\vec{M}), B_2(a)\}.$$

We have

$$\begin{aligned} \{V(\vec{M}), B_1(a)\} = & - \int_{\Sigma} -\mathcal{L}_{\vec{M}}\theta^A \wedge a \wedge *d\theta_A + \mathcal{L}_{\vec{M}}\theta^A \wedge [(a \wedge \theta^B) \wedge *_A' d\theta_B] + \\ & + \mathcal{L}_{\vec{M}}\theta^A \wedge d * (a \wedge \theta_A). \end{aligned} \quad (5.124)$$

The first term at the r.h.s. above

$$\int_{\Sigma} -\mathcal{L}_{\vec{M}}\theta^A \wedge a \wedge *d\theta_A = \int_{\Sigma} \mathcal{L}_{\vec{M}}(a \wedge \theta^A) \wedge *d\theta_A - \mathcal{L}_{\vec{M}}a \wedge \theta^A \wedge *d\theta_A$$

and the last term in (5.124)

$$\begin{aligned} \int_{\Sigma} \mathcal{L}_{\vec{M}}\theta^A \wedge d * (a \wedge \theta_A) &= \int_{\Sigma} d\mathcal{L}_{\vec{M}}\theta^A \wedge *(a \wedge \theta_A) = \int_{\Sigma} (a \wedge \theta_A) \wedge *d\mathcal{L}_{\vec{M}}\theta^A = \\ &= \int_{\Sigma} a \wedge \theta_A \wedge *\mathcal{L}_{\vec{M}}d\theta^A \end{aligned}$$

—here in the last step we used the fact that the exterior derivative  $d$  commutes with the Lie derivative  $\mathcal{L}_{\vec{M}}$ . Setting these two results to (5.124) and applying (5.122) and (5.123) we obtain

$$\begin{aligned} \{V(\vec{M}), B_1(a)\} &= - \int_{\Sigma} -\mathcal{L}_{\vec{M}}a \wedge \theta^A \wedge *d\theta_A + \mathcal{L}_{\vec{M}}(a \wedge \theta^A \wedge *d\theta_A) = \\ &= \int_{\Sigma} \mathcal{L}_{\vec{M}}a \wedge \theta^A \wedge *d\theta_A = B_1(\mathcal{L}_{\vec{M}}a). \end{aligned} \quad (5.125)$$

The other bracket

$$\begin{aligned} \{V(\vec{M}), B_2(a)\} = \int_{\Sigma} & -\mathcal{L}_{\vec{M}} p_A \wedge a \xi^A + \mathcal{L}_{\vec{M}} \theta^A \wedge \frac{1}{2} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge *(a \wedge p_D) - \\ & - \mathcal{L}_{\vec{M}} \theta^A \wedge [(*\xi^B) \wedge *'_A (a \wedge p_B)]. \end{aligned} \quad (5.126)$$

The first term at the r.h.s. above

$$-\mathcal{L}_{\vec{M}} p_A \wedge a \xi^A = -\xi^A \mathcal{L}_{\vec{M}} (a \wedge p_A) + \xi^A \mathcal{L}_{\vec{M}} a \wedge p_A = \mathcal{L}_{\vec{M}} a \wedge \xi^A p_A - (*\xi^A) \wedge * \mathcal{L}_{\vec{M}} (a \wedge p_A)$$

and the second one at the r.h.s. of (5.126)

$$\mathcal{L}_{\vec{M}} \theta^A \wedge \frac{1}{2} \epsilon^D{}_{BCA} \theta^B \wedge \theta^C \wedge *(a \wedge p_D) = -\mathcal{L}_{\vec{M}} (*\xi^D) \wedge *(a \wedge p_D).$$

Setting these two results to (5.124) and applying (5.122) and (5.123) we obtain

$$\{V(\vec{M}), B_2(a)\} = \int_{\Sigma} \mathcal{L}_{\vec{M}} a \wedge \xi^A p_A - \mathcal{L}_{\vec{M}} (a \wedge \xi^A p_A) = \int_{\Sigma} \mathcal{L}_{\vec{M}} a \wedge \xi^A p_A = B_2(\mathcal{L}_{\vec{M}} a).$$

The equation above and (5.125) give us the final result

$$\{V(\vec{M}), B(a)\} = B(\mathcal{L}_{\vec{M}} a).$$

Similarly,

$$\{V(\vec{M}), R(b)\} = R(\mathcal{L}_{\vec{M}} b).$$

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